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THE EMERGENCE OF DIMENSIONAL ANALYSIS

Dimensional analysis in 19th-century France and Great Britain

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Table of contents

Introduction

Today, dimensional analysis is one of the tools most commonly used by engineers and physicists. It consists in assigning a "dimension formula" to physical quantities, after designating a small number of physical quantities as "fundamental" (generally length, time and mass, whose dimensions are noted ,$[\mathrm{L}]$ $[\mathrm{T}]$ and $[\mathrm{M}]$). The dimensional formula for a quantity is obtained from its definition, by performing the same operations between the dimensions of the fundamental quantities as between the quantities themselves (for example, velocity has dimensions $[\mathrm{L}]/[\mathrm{T}]$). The principle of homogeneity requires that all the terms of an equation have the same dimensions. Compliance with this principle is often the first test of the validity of a theoretical result. Dimensional analysis is also sometimes used to specify the form of a law, when the related quantities and their dimensions are known in advance. These uses mask a complex conceptual situation. In particular, the link between dimensions and the nature of physical quantities has often been the subject of disagreement. Here, we take a historical approach, while emphasizing what concerns the nature of physical theories.

We will limit our historical investigation to the 19th century, from the foundations laid by Joseph Fourier in the 1820s to Aimé Vaschy's theorem in 1892. Few studies have been devoted to the history of dimensional analysis, even in this founding period. There are two articles on the subject in the *Journal of the Franklin Institute*, a publication devoted to engineering and applied mathematics. The second, by Roberto Martins,[1] answers the first, due to Enzo Macagno.[2] Martins admits:

> A complete history of dimensional analysis has yet to appear. Several authors have complained about this gap on occasion, but to my knowledge only one publication now exists on the subject: Macagno's review published in this Journal.[3]

Macagno's study dates from 1971 and covers a vast period. It stretches from Greek antiquity to the first half of the 20th century, around 1830. Macagno devotes the first three pages of his article to pre-Fourier developments. One can guess the reason for his interest in a period that others would have dismissed as prehistory rather than history of the subject: he intends to show the close link between conceptions of homogeneity and the possibility of mathematical physics in the modern sense. He sees certain ancient conceptions of homogeneity as the major obstacle to the development of mathematical physics:

> Most branches of physics, with the exception of statics and hydrostatics, were not developed into exact sciences by the Greeks in the same way as geometry and astronomy. If, for example, the product of four lengths was considered an aberration, how could velocity - as a quotient of quantities of different types - stand any chance of being conceived? The derived quantities with which we are so familiar today were not recognized as such, nor were they derived as entities in their own right.[4]

The demand for homogeneity as conceived in the Greek world made the very entities that make up modern physics inconceivable.

[1] Martins 1981: 331-337.
[2] Macagno 1971: 391-402.
[3] "The complete history of dimensional analysis has not yet been written. Several authors have at one time or another complained of this vacancy, but to this author's knowledge only one paper has been issued hitherto on this subject: Macagno's review published in this Journal." Martins 1981: 331.
[4] "Most branches of physics, with the exception of statics and hydrostatics, were not developed under Greek leadership into as exact sciences as were geometry and astronomy. One of the obstacles was probably the reluctance to perform unorthodox operations with physical quantities; if for instance, the product of four lengths was considered an aberration, how could velocity - as a ratio of quantities of different species - even have a chance? Derived quantities, so familiar to us nowadays, were not recognized as such and were never derived in an independent form." Macagno 1971: 391-392.

In the following two pages, Macagno outlines the evolution of the concepts of homogeneity and dimensions in the Middle Ages, the Renaissance, and the 18th and 19th centuries, leading to the gradual appearance of derived quantities and physical equations. His analysis is often based on specific remarks made by the scientists he discusses.

Macagno devotes two further pages to Fourier's seminal contributions in the 1820s, which he sees as marking the beginnings of dimensional analysis as such:

> It fell to Fourier to formulate the principles of dimensional analysis; Fourier [...] established the foundations of dimensional analysis.[5]

We'll discuss his views in more detail in Chapter 1. After Fourier, he saw no significant progress until the work of Rayleigh :

> After Fourier, there were no developments in dimensional analysis for half a century.[6]

Macagno then turns to the dimensional method developed by Lord Rayleigh in the 1870s. He makes no mention of the blue sky study, concentrating instead on the treatise *Theory of sound*, no doubt because the British scientist presented a systematic treatment of his approach. Macagno sees in this text "the beginning of the application of the principles enunciated by Fourier more than fifty years earlier."[7]

In addition to Rayleigh's contribution, the second half of the 19th century was characterized for Macagno by the work of French telegraph engineers, especially Emmanuel Carvallo and Aimé Vaschy, culminating in Vaschy's theorem in 1892. It should be noted that Macagno makes no mention of the joint publications by Vaschy and Ernest

[5] "It was reserved for [Fourier] to formulate the principles of dimensional analysis; Fourier [...] established the foundations of dimensional analysis." Macagno 1971: 393.
[6] "After Fourier, no important development in dimensional analysis took place for half a century." Macagno 1971: 395.
[7] "... the beginning of the application of principles enunciated by Fourier more than fifty years before." Macagno 1971: 395.

Mercadier, which, as we shall see in chapter 6, are of great interest in terms of the very conception of dimensions, and respond to the concerns encountered by British scientists in the context of electrical and magnetic phenomena.

In the last third of his publication, Macagno describes the rediscovery of this theorem by Dimitri Riabouchinsky in 1911, an interesting controversy between the latter and Rayleigh in 1915, and new versions of Vaschy's theorem by Edgar Buckingham and Percy Bridgman.[8]

Macagno chooses not to deal with the question of similarity, although he does mention its importance:

> To include in this article more than is strictly necessary about similarity would run counter to our attempt to focus our attention on dimensional analysis and disentangle the two subjects as far as possible.[9]

Martins' article appeared ten years later, in 1981. Its purpose is to criticize Macagno's views on certain points. In particular, Martins disagrees with his analysis of René Descartes' conceptions on the subject of dimensions:

> In this discussion of Macagno's work, we have attempted to illustrate some of the dangers of the influence of implicit assumptions in historical research. In looking for precursor concepts to Fourier's, Macagno focuses his attention on earlier uses of the *term* dimension (for example, in Descartes), without having analyzed the properties of the concept in order to be able to identify the prior uses of those properties that constitute the essence of the concept.[10]

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[8] Macagno 1971: 398-401.

[9] "To include more than is strictly necessary about similarity in this article would defeat the purpose of centering the focus on dimensional analysis and of trying to disentangle the two subjects as much as possible." Macagno 1971: 395.

[10] "In this discussion of Macagno's work, we have tried to illustrate some of the dangers of the influence of implicit assumptions in historical research. In looking for antecedents of Fourier's concept, Macagno drew attention to earlier uses of the *word* dimension (such as Descartes'), without analysing the properties of the concept in order to look for the previous uses of these properties that comprise the essence of the concept." Martins 1981: 335.

Moreover, Martins argues that dimensional analysis was employed several decades before Fourier's work, and concludes:

> Macagno implicitly assumes that dimensional analysis could not have appeared prior to the establishment of the concept of physical dimensions, and consequently looks only for instances of its use subsequent to Fourier's work. But it is common in the history of science to find an idea used obscurely before it is explicitly defined. As we have shown, this was the case with dimensional analysis.[11]

In fact, as we shall see and as Martins' article itself shows, such uses, while real, were extremely limited.

Martins' work is by no means a general history of dimensional analysis. In particular, it does not deal with the period after Fourier, which is of particular interest to us.

It was some thirty years before we again found a general historical study of dimensions. In 1998, John Roche devoted two chapters to the subject in his treatise *The mathematics of measurement*, which constitutes the most in-depth discussion of the subject.[12] He deals with dimensional analysis *stricto sensu* in Chapter 10, entitled "Dimensional analysis and the quantity calculus," and reserves the following chapter, "Dimensional Exploration," for the use of homogeneity considerations in innovative perspectives, aimed at generating new knowledge.

Unlike Macagno, Roche does not deal with ancient conceptions, choosing to begin his study with the Renaissance. He devotes the first two pages of his study to the contributions of François Viète, Descartes and John Wallis, analyzing how their

[11] "Macagno implicitly assumed that dimensional analysis could not have arisen before the establishment of the concept of physical dimensions, and therefore only looked for instances of dimensional analysis subsequent to Fourier's work. But it is a common phenomenon in the history of science that an obscure idea is used before being explicitly defined. As has been shown, this was the case in dimensional analysis." Martins 1981: 336.

[12] Roche 1998: 188-218.

conceptions of dimensions generalized the concrete geometric notion of this term into a more abstract one, as we shall see in chapter 1 of this study. We can already note that, as far as Descartes' views are concerned, Roche is closer to Macagno than to Martins - if only because he sees Descartes' use of the term as relevant to the history of the subject.[13]

Roche then turns to Fourier's contributions, analyzing them in greater detail than Macagno had done.[14] For him, dimensional analysis is far from irrelevant between Fourier and Rayleigh. He describes the intermediate contributions of Carl Friedrich Gauss, Wilhelm Weber and James Clerk Maxwell.[15] A relatively large part of his discussion concerns the notations of these authors. The reason for this is that they are intimately linked to the concern to show dimensions in equations:

> The decision in the early 19th century by the French School of Analytical Physics to interpret the equations that express the mathematical laws of physics as relationships between abstract numerical measurements seems to have led to the attempt, by Fourier and Gauss in particular, to create an alternative notation that could once again relate to physical quantities.[16]

This is what Roche refers to when he speaks of "*quantity calculus*." As we shall see in Chapters 2 and 3 of this work, Gauss inaugurated this new calculus, followed by Weber and then Maxwell. [17]

Roche's treatment of Maxwell's contributions touches on one of the most interesting aspects of the history of dimensional analysis. Roche emphasizes Maxwell's mechanistic view that electrical and magnetic phenomena are fundamentally mechanical in nature. This,

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[13] Roche 1998: 188-190.
[14] Roche 1998: 190-197. Again, we'll see in what terms in Chapter 1.
[15] Roche 1998: 197-205.
[16] "The decision by the French school of analytical physics at the beginning of the nineteenth century to interpret all of the equations expressing the mathematical laws of physics as relations between abstracted measuring numbers seems to have led to an attempt, particularly by Fourier and Gauss, to create an alternative notation which could again refer to physical quantities." Roche 1998: 197.
[17] Roche 1998: 200-202.

he argues, prompted Maxwell and many of his successors to conceive of dimensions as reflecting the nature of physical quantities. As we shall see, this conception is opposed to that initiated by Fourier, where dimensions are defined by considerations of invariance under changing units. And this opposition will be one of the major themes of the present study. We will discuss Maxwell's contributions in Chapter 3, and his dimensional thinking in Chapters 6 and 10. It may be noted that, like Macagno, Roche does not deal with the discussion between Mercadier and Vaschy, which embodies this "strong" conception of dimensions. This discussion is the subject of our chapter 6.[18]

Roche concludes Chapter 10 with the contributions of Campbell and Bridgman and the subsequent status of dimensional analysis.[19]

In Chapter 11, *Dimensional Exploration*, Roche first devotes a few pages to Rayleigh's contributions and Vaschy's theorem. Like Macagno before him, he concentrates on the *Theory of sound* and does not deal with Rayleigh's earlier work.[20] The rest of the chapter deals with later developments in these approaches, in particular the realization of their limitations. In this context, he again discusses the work of Norman Campbell (1880-1949) and Bridgman.[21] Roche also gives a brief discussion of the problem of choosing fundamental units, and of the problems encountered in electricity and magnetism. We will develop these themes mainly in Chapters 4 and 5 of the present work.[22]

The present study differs from the previous ones in that it concentrates on the post-Fourier period, stopping essentially at the theorem formulated by Vaschy in 1892. In

[18] Roche uses the expression ""strong view" of dimensions" to refer to this. Roche 1998: 202-205.
[19] Roche 1998: 205-207.
[20] Roche 1998: 208-211.
[21] Roche 1998: 212-217.
[22] Roche 1998: 214-215.

addition, it seeks to identify the varying motivations behind work involving dimensional analysis throughout the century. In particular, we'll see the importance of the rise of electromagnetism and its application to the telegraph. Several of the protagonists of dimensional analysis were telegraph engineers - Aimé Vaschy in particular. In addition, our study highlights the profound but often unconscious disagreements about the link between dimensions and the nature of physical quantities. We'll be coming back to this link frequently.

The pre-Fourier period and Fourier's own contributions are covered in Chapter 1. Chapter 2 is devoted to the work of Gauss and Weber. Although they are not explicit examples of dimensional analysis, they are based on similar considerations and come close to Fourier's treatment. Chapter 3 is a study of the contributions of Maxwell and some of his co-authors, which can be seen as a synthesis between the work of Fourier on the one hand, and Gauss and Weber on the other. Chapter 4 focuses on the problem of choosing the fundamental quantities used to express the dimensional formula. As we shall see, several studies have been devoted to this question.

One of the most active branches of physics at this time, electromagnetism, is the context for Gauss's, Weber's and Maxwell's work on dimensions. It presents particular difficulties for dimensional analysis. These were most acutely expressed in the controversy over the dimensions of the magnetic pole, which took place mainly in Great Britain. This controversy is the subject of Chapter 5. The difficulties encountered in electromagnetism also gave rise to an effort at clarification by two French engineers, Mercadier and Vaschy. Their attempt broke with the tradition established by Fourier, as we shall see in chapter 6.

The next three chapters are devoted to what Roche would call dimensional explorations: Chapter 7 deals with Rayleigh's contributions, the next concentrates on similarity, and Chapter 8 deals with the development of Vaschy's theorem. Finally, the tenth and last chapter is a return to the problem of the physical meaning of dimensions. As we have already said, this question is a common thread running through our work. The notion that the dimensions of a physical quantity reflect its nature is the motivation behind the principle of homogeneity before Fourier. By defining the dimensions of a quantity through the invariance of an equation under a change of units, Fourier broke the necessary link between dimensions and the physical nature of a quantity. The tension between the two currents of thought would be present throughout the 19th century and beyond.

Chapter 1

The concept of dimension from its origins to Fourier

1. 1 - The dawn of the concept of dimension

It's not until Joseph Fourier that we can really start talking about dimensional analysis. However, there were precursory concepts and considerations well before Fourier. These can be considered as such because they respond to a concern for homogeneity, i.e. the examination of the conditions of possibility of various operations between quantities according to their nature (for example, quantities of different natures cannot be added). In the context of work prior to Fourier, we will therefore use the term "dimension" in the sense of "that which must be similar in different terms in order for them to satisfy the principle of homogeneity." The concept of dimension may therefore differ from one author to another - and how it differs in this way is precisely one of the issues that interests us. For the first authors, it's clear that the concept of dimension merges with that of spatial dimension - which explains the current polysemy of the word. In other words, concerns about homogeneity originally concerned spatial dimensions. Let's retrace the stages through which the term's meaning was broadened.

This concern for homogeneity can be traced back to Greek antiquity. In his *Elements*, Euclid (~325-~ 270 B.C.) stresses the need for certain operations to be performed on geometric quantities of the same kind. These geometric quantities - length, area and volume - were the only physical quantities discussed at the time. What's more, area and volume can only be processed by operations that are themselves geometric. Indeed, while there was a concept of units for lengths, this had not yet been extended to areas and volumes. It is therefore impossible to express the measurement of these two quantities by a numerical value followed by a unit and, consequently, to perform arithmetic operations on such numerical values. In particular, Euclid does not provide equations for arithmetically calculating the area or volume of geometric figures. We do, however, find operations on these quantities, which constitute the geometric counterpart of addition (and therefore also of subtraction), multiplication and division. In the case of addition, all you need to do is juxtapose several quantities. It is therefore, of course, carried out between homogeneous quantities, and gives a homogeneous result. As for multiplication, there are two distinct forms. The first is simply the repetition of addition operations, juxtaposing the same quantity several times. A geometric quantity is multiplied by a whole number, and the result is the same as the geometric quantity. The second form of multiplication enables us to obtain quantities that are heterogeneous to those we're working on: thus, the product of two lengths produces a rectangle, and the product of three lengths a parallelepiped. The most interesting case is that of division, as it is undoubtedly the one that most directly reflects a concern for homogeneity. In Book Five of his *Elements*, Euclid presents his theory of ratios, probably due to Eudoxius of Cnidus. In it, he presents a series of lemmas concerning relationships between quantities. The question of homogeneity is immediately present; he poses the need for the quantities concerned to be homogeneous in his very definition of ratio: "A ratio is a type of relationship, concerning size, between two quantities of the same kind." [1]

It should be stressed that such ratios are not conceived as numbers. Indeed, in antiquity, the only numbers recognized as such were integers - and, at the end of the classical period, rationals. Irrationals, on the other hand, are not thought of as numbers.[2] Yet the (homogeneous) quantities that enter into ratios are arbitrary, and can perfectly well be incommensurable: in antiquity, their ratio cannot therefore be thought of as a number. These considerations are particularly evident in a commentary by Proclus (411-485 A.D.), who states:

> The fact that a ratio is always rational belongs to arithmetic alone, and not entirely to geometry, since irrational ratios also exist in the latter... Arithmetic is the first to consider things commensurable... Geometry considers them after taking arithmetic as its model, and thereby establishes that commensurable things are all those that have between them the relationship of a number to a number, because commensurability exists primordially in numbers.[3]

At the same time as requiring ratios to be taken between homogeneous quantities, the theory of Eudoxus and Euclid provides a means of comparing heterogeneous quantities, in that Euclid also defines the concept of proportion: "Let us call 'proportional' those quantities which have the same ratio."[4] Thus, if we have four quantities A, B, C and D such that :

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[1] Greek Mathematics, 445 lemma 3; greek text 444: "γ'. Λόγος έστι δύο μεγεθων όμογενων ή κατα πηλικότητά ποια σχέσις."

[2] What is commonly regarded as the Pythagoreans' discovery of the irrationality of $\sqrt{2}$ is conceived in terms of the incommensurable nature of two *geometric* quantities: the ratio of the diagonal of a square to its side cannot be expressed by a ratio of numbers (implied to be integers).

[3] Proclus de Lycie, "Les commentaires sur le premier livre des Eléments d'Euclide," prologue 51, Desclées de Brouwer, 1948.

[4] "ς'. Τα δε των αυτον εχοντα λόγον μεγέθη ανάλογον καλείσθω." Euclid, *The Elements*, Book V, Lemma 6.

$$\frac{A}{B}=\frac{C}{D} \tag{1}$$

A and B must be homogeneous with each other, and C and D must be homogeneous with each other, but A and B are not necessarily homogeneous with C and D.

Such "double proportions" (usually noted as *A:B :: C:D*) were long used to express relationships between physical quantities. This was a way of getting around the difficulty imposed by the principle of homogeneity; the operations themselves had to be performed between homogeneous quantities. The Greek mathematicians' ideal of rigor in this respect was met. Taking the ratio of quantities of different natures seemed to them as aberrant as trying to add them together. For example, according to François Viète, Adraste (2nd century CE) declared that it was inconceivable how heterogeneous terms could influence each other.[5] In such a context, the development of true equations between heterogeneous physical quantities is impossible, as is the concept of derived quantities, as Enzo Macagno notes.[6]

As we shall see, this situation persisted at least until the Renaissance. It is conceivable that it was favored by the extraordinary diversity of measurement units, which made it inconvenient to express relationships between physical quantities in the form of equations, since some of the numerical factors contained in these then varied from one system to another.

At the same time, a number of works directly related to the concept of dimensions can be found as far back as antiquity. Although there's nothing to suggest that this work was related to considerations of homogeneity, it is worth noting because of the aforementioned historical link between the two concepts of dimension.

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[5] François Viète 1630: (translation by Vasset de Viète 1591), 7.
[6] Macagno 1971: 391-392.

In his *Treatise on Heaven*, Aristotle (384-322 B.C.) defends the existence of three spatial dimensions:

> A magnitude so divisible in one way is a line, so in two ways a surface, and so in three a solid. Beyond that there are no other magnitudes, for three dimensions constitute all that exists, and what is divisible in three dimensions is divisible in all.[7]

Aristotle's discussion of divisibility already suggests a preoccupation with measurement. He also seems partly motivated by a fascination with the number three, inherited from the Pythagoreans.[8] Ptolemy (ca.90-ca.168 CE) seems to have gone a step further, attempting to prove that there can only be three spatial dimensions, in a short treatise devoted entirely to the subject.[9] Pappus (ca.290-340 CE), for his part, refers to contemporary authors who took the liberty of speaking of a rectangle multiplied by a square, without even bothering to explain what they meant by this.[10] Enzo Macagno suggests that he was referring to the derivation of a formula for the area of a triangle, involving the product of four lengths.[11] In any case, it would seem that some Greek mathematicians did use four-dimensional quantities in their calculations. However, such audacity does not seem to have been commonplace, quite the contrary: in the example we have just given, Pappus alludes to this approach only to criticize it.

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[7] Aristotle, *Treatise on Heaven*, beginning of Book 1 (~line 15).

[8] Aristotle, *Treatise on Heaven*, beginning of Book 1 (~line 25).

[9] This treatise, entitled *Des distances,πε ρ ιδια στα σεως* , has not survived. Nevertheless, we know the essence of Ptolemy's reasoning from Simplicius (6th century CE) in his *Commentaries*: "The admirable Ptolemy in his book "Of Distances "has well proved that there are only three dimensions, from the necessity that dimensions must be defined, and that defined distances must be taken along perpendicular lines, and because it is only possible to take three lines perpendicular to each other, from which the plane is defined and the third measuring depth; in such a way that if there were another dimension after the third, it would be entirely without measure and without definition." *Simplicii in Aristotelis De Caelo Commentaria*, ed. Heiberg, 1894, 33. Translated from English quotation *in* Manning 1914: 1.

[10] Heath 1921b: 402.

[11] Macagno 1971: 391.

It's perfectly acceptable to obtain a rectangle by operating geometrically on two lengths, but such heterogeneity in the result of an operation is not extended to non-geometric physical quantities. As far as surviving sources indicate, there was no concept of derived quantities in antiquity. We can certainly find concepts that correspond to physical quantities subsequently considered as such (for example, Heron of Alexandria's notion of the moment of a force),[12] but these quantities were not considered as such at the time.

It's only in the Middle Ages that we find the first traces of the notion of derived magnitude, and even then we have to be very careful. Some authors - such as Jean de Muris (1290-1351) - take the product or quotient of quantities that refer to magnitudes of different natures, while expressing the matter in such a way as to respect the ancient principle of homogeneity as far as possible.[13] For them, it's not a question of operations on different quantities, but on numbers representing these quantities:

> Multiply (the excess weight) by the number representing the total length of the beam, and then divide this product by the number representing twice the length of the other bar, and what we get is the number representing the weight that...[14]

Such attempts are made without at the same time finding attempts to extend the concept of dimension, since here again the approach is to circumvent the principle of homogeneity rather than redefine it.

1. 2 - François Viète: the introduction of dimensions greater than three

[12] Heron of Alexandria, "Les mécaniques ou l'élévateur," translated into French by Carra de Vaux, *Journal asiatique*, 1894, ser.9 T.4, 25-29. See Duhem 1906: 304.
[13] Macagno 1971: 392.
[14] "multiply (the excess weight) by the number representing the length of the whole beam, and then divide this product by the number representing twice the length of the shorter arm, and what results is the number representing the weight which" quoted from Macagno 1971: 392.

It was undoubtedly with François Viète's (1540-1603) treatise *L'Algèbre nouvelle* that the concept of dimension, in the sense we have defined it, began to be extended. Viète's concern for homogeneity led him to define real rules of a numerical nature, and thus the rudiments of dimensional analysis. [15]

In general, Viète was interested in methods for solving equations of the second, third and fourth degrees. He is credited with being one of the first to introduce algebraic notation: he represents quantities by letters, distinguishing unknowns from those assumed to be known. He designated the former by vowels and the latter by consonants. Although some of his methods for solving equations are reminiscent of the work of Sharaf al-Din al-Tusi, it would seem that Viète was not directly influenced by Arab mathematicians, but by Cardan, as well as by the Greeks, notably Diophantus. In the mathematical context of his time, equation solving was still based on a geometric approach. Unlike de Muris, who managed to avoid it, Viète was faced with the problem of homogeneity. He couldn't deal with equations such as $x^3 + x^2 = 1$, as this would be tantamount to adding a volume to an area, which makes no sense. Unknown quantities must therefore be multiplied by known quantities so that the terms have the same dimension, for example: $A^3 + B\,A^2 = B^2 Z$

For these reasons, Viète devotes an important part of *L'Algèbre nouvelle* to the principle of homogeneity, which he states in the form that only homogeneous terms can be compared with each other.[16] He is only interested in geometric quantities, and what he means by "homogeneous" is that they have the same spatial dimensions.

[15] *L'Algèbre nouvelle* (1630, Paris) is the French translation by A. Vasset of *In artem analyticam isagoge*, (1591). In English, the same work is entitled *The Analytic Art.*
[16] Viète 1630: 7.

Viète covers some of the rules that were common in Antiquity. Thus, when two quantities are added or subtracted, they must be homogeneous, and the result must also be homogeneous. He is more original in the case of multiplication and division, which are more complex. Although Viète does not provide a precise definition of these operations, it seems clear that he treats the multiplication of any spatial quantities on the model of the passage from two lengths to an area:

> That there are two sizes A. & B. One must be multiplied by the other.
> Therefore, since it is necessary to multiply one magnitude by another magnitude, they will produce by their multiplication a magnitude that will be heterogeneous to them, i.e. of different kinds, and hence the one produced will be conveniently signified by the word (by or under) as A. by b. or [...] that such a magnitude is made under A. and B. and this if A. and B. are simple lengths or widths.[17]

He states that the product of two quantities is heterogeneous to both.[18] As for division, he does not hesitate to carry it out between two heterogeneous quantities - obtaining a result heterogeneous to both. However, this should not be seen as a rejection of Euclid's theory of ratios: the division in question is merely the inverse operation of multiplication, which generates areas from lengths, and not strictly speaking a ratio:

> So let A. be a length, B. a plane, from which a small line will be drawn between B. plane, the highest [i.e. the highest term], & A. the lowest, which is by which the division is made.[19]

When it comes to the degree of the result, Viète couldn't be more explicit, systematically setting out some twenty different cases. He multiplies a width by a length to obtain a plane,

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[17] "Qu'il y ait deux grandeurs A. & B. Il faut multiplier l'vne par l'autre.
Doncques puis qu'il faut multiplier vne grandeur par vne autre grandeur, elles produiront par leur multiplication vne grandeur qui leur ſera Heterogene, c'eſt à dire de diuers genre, & partant celle qui ſera produicte, ſera commodément ſignifiée par le mot (par ou ſoubs) comme A. par B. ou comme quelqu'autres ont en pour aggreable, que telle grandeur eſt faite ſoubs A. & B. & vela ſi A. & B. ſont ſimples longueurs ou largeurs. "Viète 1630 : 13.

[18] Viète 1630: 13.

[19] "Soit donc A. vne longueur, B. vn plan, partant l'on tirera vne petite ligne entre B. plan, la plus haute, & A. la plus baſſe, qui eſt celle à laquelle ſe fait l'application." Viète 1630 : 16-17.

then a width by a plane to obtain a solid, and so on.[20] He treats divisions in the same way. We have, for example:

> That if we pose that B. is cube, A. plane, by $\frac{\text{B.cube}}{\text{A.plan}}$ we will give the width that comes from the division of B. cube by A. plane. And if we pose that B. is cube, A. length, by $\frac{\text{B.cube}}{\text{A}}$ we will represent the plane that results from the division of B. cube by A....[21]

It is in this context that Viète introduces what is undoubtedly one of the most interesting aspects of his work: since multiplication quickly gives rise to magnitudes of "dimension" greater than 3, he generalizes the concept of geometric magnitude. He thus defined, after the solid, the "plane plane," the "solid plane," and so on up to the "solid solid":

> The width of the solid makes the plane plane...
> The solid [multiplied] by the solid plane makes the solid plane solid.[22]

Viète seems to have inaugurated this type of generalization, apart from the above-mentioned authors criticized by Pappus.

In short, Viète attempts to formalize the principle of homogeneity in a precise manner. What's more, he implicitly goes beyond the principle of homogeneity. *Strictly speaking,* the latter only applies to addition and subtraction. For multiplication and addition, neither the terms on which the operations are performed, nor the results, are homogeneous. The originality of Viète's work lies in the fact that he lays down precise

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[20] Viète 1630: 15-16.

[21] "Que ƒi on poƒe que B. ƒoit cube, A. plan, par $\frac{\text{B.cube}}{\text{A.plan}}$ l'on donnera la largeur qui vient de l'application de B.cube, à A. plan. Et ƒi on poƒe que B. soit cube, A. longueur, par $\frac{\text{B.cube}}{\text{A.}}$ l'on repreƒentera le plan qui reƒulte de l'application de B. cube à A. […];"Viète 1630 : 17. The Latin original presents these "fractions" as we have presented them in the text.

[22] "The width by theƒ olide faut le plan plan..., Leƒ olide [multiplié] par le planƒ olide fait le planƒ olideƒ olide." Viète 1630: 15-16.

rules concerning this heterogeneous character, developing a terminology for this purpose. What's more, although Viète's concept of dimensions remains purely spatial, it is extended to dimensions greater than three. This is the first time, as far as we can tell, that the idea of spatial dimensions beyond three appears in the context of considerations of homogeneity.

1. 3 - 17th and 18th centuries: Extension of the concept of dimension to non-spatial quantities

The concept of dimension was then extended in the 17th century, when it came to designate non-spatial quantities. It was used by two of the period's authors: René Descartes (1596-1650) in the first half of the century, and John Wallis (1616-1703) in the second.

Around 1628, Descartes declared:

> By dimension we mean nothing other than the mode and manner according to which any object is considered measurable; so that not only length, width and depth are dimensions of bodies, but also gravity is the dimension according to which objects are weighed; speed, the dimension of motion: and so on. The division itself into several equal parts, whether real or intellectual, is properly the dimension according to which we count things.[23]

Descartes therefore uses the term dimensions in the sense of "measurable physical magnitude," whether or not it's a spatial magnitude.

Whether or not he felt that an object has a precise number of dimensions is unclear. It would seem not:

> In one and the same object there may be infinitely diverse dimensions, which add absolutely nothing to the things that possess them, but which must be understood in the same way, whether they have a real foundation in the objects themselves, or whether they have been arbitrarily invented by our minds. Indeed, the weight of a body, the speed of motion, or the division of the century into years and days are real things: but the division of the day into hours and moments is not.[24]

[23] Descartes 1628: in *Descartes 1824-1826:* 305.
[24] Descartes 1628: in *Descartes 1824-1826:* 306.

When it comes to geometric figures, on the other hand, Descartes declares:

> In a triangle, when you want to measure it exactly, you need to know three things about the side of the object, i.e. the three sides, or two sides and an angle, or two angles and the area, etc.; similarly, in a trapezoid you need five pieces of data, six in a tetrahedron, and so on. All this can be called dimensions.

Admittedly, these considerations do not seem to stem from a concern for homogeneity. Roberto Martins believes that Descartes' conceptions have no real place in a history of dimensional analysis. With reference to Macagno's earlier study:

> In looking for precursor concepts to Fourier's, Macagno focuses on earlier uses of the *term* dimension (such as in Descartes), without having analyzed the properties of the concept in order to be able to identify the prior uses of these properties that constitute the essence of the concept.[25]

However, Descartes sometimes seems to use the concept of dimension to differentiate between physical quantities, following the implicit principle that the same quantity can only have a given number of dimensions:

> I have spoken of the force that serves to lift a weight to some height, *which force always has two dimensions*, and not of that which serves at each point to support it, *which never has more than one dimension*, so that these two forces differ as much from each other *as an area differs from a line*. For the same force that a nail must have to support a weight of 100 pounds for a moment, is also sufficient to support it for a year, provided it does not diminish. But the same quantity of this force that serves to lift this weight to the height of one foot, is not sufficient *cadem numero* to lift it to the height of two feet; & it is no more clear that two & two make four, than it is clear that double must be employed. [26]

[25] "In looking for antecedents of Fourier's concept, Macagno drew attention to earlier uses of the *word* dimension (such as Descartes'), without analysing the properties of the concept in order to look for the previous uses of these properties that comprise the essence of the concept." Martins 1981: 335.
[26] Descartes to Mersenne 1638: 134.

Martins interprets this type of usage as showing that Descartes understood the term dimension simply in the sense of *magnitude*.[27] The fact remains that Descartes associated the number of dimensions with the nature of physical quantities.

Descartes is referring here to what we would today call a force and an impulse. Although he uses the same term force for both, he insists that they must be distinguished on the basis of their dimensions, defined as the type of measurements needed to characterize them completely (like the geometric figures mentioned above): i.e., a force when it comes to sustaining, a force and a time when it comes to lifting. Such a conception is implicitly close to considerations of homogeneity in that it relates to the *nature of* a physical quantity. But we must be wary of seeing it as a conscious allusion to homogeneity. Descartes does not use his concept of dimension in the context of the homogeneity of formulas. There's a simple reason for this: whereas in the Middle Ages a de Muris insisted that calculations should be thought of as taking place on numerical values, Descartes translates equations (he uses the term) in entirely geometrical terms. In his *Géométrie*, he describes how to solve equations by representing quantities as line segments, and translating the various operations into geometric operations on these line segments:

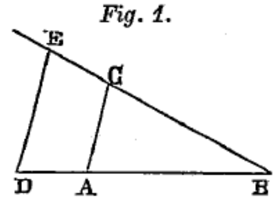
> All geometric problems can be easily reduced to such terms, that it is only necessary to know the length of a few straight lines in order to construct them.
> And as the whole of arithmetic is composed of only four or five operations, which are addition, subtraction, multiplication, division, and the extraction of roots, which can be taken as a kind of division, so in geometry there is nothing else to do with the lines we are looking for to prepare them to be known, than to add others to them, or remove some.[28]

For example, according to Thales' theorem, a multiplication is performed as follows:[29]

[27] Martins 1981: 334.
[28] Descartes 1637: 1.
[29] Descartes 1637: 2.

La multiplication. Soit, par exemple, AB *(fig. 1)* l'unité, et qu'il faille multiplier BD par BC,

Fig. 1.



je n'ai qu'à joindre les points A et C, puis tirer DE parallèle à CA, et BE est le produit de cette multiplication.

As a result, the homogeneity principle is automatically satisfied for calculation purposes: all the quantities on which operations are performed are line segments. [30]

In 1684, John Wallis in turn used a concept of dimension extended to non-spatial quantities:

> A solid can also take on additional magnitudes (certainly not an additional spatial extension). For example, this solid can have a Weight (which influences each part of the Solid in the same way).... And, when considered in this way, in addition to the three LoLa E dimensions (length, width, thickness) it has acquired a fourth (albeit non-spatial) dimension of Weight.... And if to these four dimensions we add the fifth of celerity:la Force which results LoLa E P C, is a five-dimensional quantity. [31]

Although he makes no reference to Descartes, it's conceivable that Wallis was inspired by him, as he mentions precisely the same quantities as Descartes, i.e. weight and speed (or "celerity").

Like Viète, Wallis uses dimensional arithmetic in the sense that, given the dimensions of two quantities and an operation between them, he obtains the dimensions of the result. He ventures further than Viète in such calculations, in that he introduces the idea

[30] That said, Descartes explicitly brings in the concept of measurement (remember that it was already present in Aristotle's work) as soon as he extends the notion of dimension to non-spatial quantities.

[31] "a solid may further admit (tho not another Local extension, yet) a Superfetation of more Magnitudes. As, for instance, this Solid is capable of Weight, (which may equally influence each part of the Solid,... And, when so considered, besides the three dimensions LBT (Length, Breadth and Thickness) it has acquired a fourth Dimension (though not Local) of Weight... And if to those four dimensions we superinduce the fifth of Celerity: the Force hence arising LBTWC, is a magnitude of five Dimensions." Quoted in Roche 1998: 190; Wallis 1684: 94-95; 78.

that quantities can have zero or even negative dimensions. In fact, he states: "when a species is divided by another of the same dimensions, a quantity of zero dimensions results.[32] As for negative dimensions, he considers elsewhere four quantities of dimension 1, denoted *A*, *B*, *C* and *D*, and explains that the quantities :

$$\frac{A}{B \cdot C} \quad \textbf{and} \quad \frac{A}{B \cdot C \cdot D} \qquad \textbf{(2)}$$

are respectively -1 and -2 , "because $1-2=-1$ and $1-3=-2$."[33] Wallis thus uses the same rules as Viète, namely that when we multiply quantities, we must add their dimensions, and when we divide quantities, we must subtract their dimensions.

It's interesting to note that Wallis, unlike Viète, takes care to define what he means by multiplication, and to distinguish between two kinds (in fact using the term for only one of them) that correspond to the distinction we've already mentioned in antiquity:

> Multiplying only gives a new Proportion, not a new Dimension: as when a mile is multiplied by 100, it becomes100 miles ; which is still only a length, it has nothing of Width, or surface magnitude; and this whatever it is Multiplied by will never make an Acre. But *ducere*, to draw in a magnitude, does not give a new Proportion of a Magnitude of the same nature, but gives a new Dimension, and consequently creates a Magnitude of another nature (Being as it were an accumulation of one Magnitude upon another). As when a length drawn into a Length, produces a Width[34] ...which is a heterogeneous quantity (of another kind of nature than this line...).[35]

[32] "...where one species is divided by another of the same dimensions a quantity of null dimensions results." Quoted in Roche 1998: 190; Wallis 1695: 103.
[33] Cited by Roche 1998: 190; Wallis 1695: 103.
[34] It seems reasonable to assume that Wallis means a size with width as well as length, and therefore a plane.
[35] "to *multiply* gives only a new proportion, not a new Dimension: As when a Mile is multiplied by 100, becomes100 miles ; which is but length still, it hath nothing of Breadth, or superficial magnitude; and this however Multiplied will never make anAcre . But *ducere*, to draw into a Magnitude, doth not give a new Proportion of the same kind of Magnitude; but gives a new Dimension, and thereby creates a Magnitude of another kind (Being as it were a superfetation of one Magnitude upon another). As when Length drawn into Length, produceth Breadth,... which is a heterogeneous quantity (of another kind in nature) to that line" Quoted by Roche 1998: 189-190; Wallis 1684: 94-95; 78.

Wallis therefore reserves the term multiplication for the operation of taking *n* times a physical quantity, where *n* is an integer. When it comes to "multiplying" physical quantities together, he uses another term, "*ducere*."

Equally important for the extension of the term dimension is the fact that the period saw the appearance of the concept of derived quantities, due to Newton. In his *Principia*, Newton states:

> I call a *genitum* any quantity that is not obtained by addition or subtraction of various parts, but is generated or produced in arithmetic by multiplication, division, or extraction of the root of any term...[36]

By the end of the 17th century, the principle of homogeneity was no longer confined to addition and subtraction. Multiplying or dividing mutually heterogeneous quantities (or taking the root of a quantity) became legitimate operations, on the understanding that other quantities, heterogeneous to the first, were obtained. This was already the case with Viète, but then it only applied to geometric and spatial quantities (length, area, etc.). A century later, it was generalized to all physical quantities, as Newton emphasized. At the same time, the term "dimension" was also extended to non-spatial quantities, as early as the first half of the century - although its relationship with considerations of homogeneity is debatable. As John Roche puts it:

> The 17th century established a concern for geometric homogeneity in the equations of mathematical physics and made several attempts to generalize the concept of dimension, both conceptually and formally.[37]

[36] "I call any quantity a *genitum* which is not made by addition or subtraction of various parts, but it is generated or produced in arithmetic by multiplication, division, or extraction of the root of any term whatsoever" Newton 1687: Book 2, lemma 2.

[37] "The seventeenth century established a concern for geometrical homogeneity in the equations of mathematical physics and made various attempts to generalise the concept of dimension, both conceptual and formal," Roche 1998: 190.

On the other hand, there were relatively few innovations on the homogeneity problem in the 18th century. The concept of dimension as defined above continued to be used, and that of zero dimension seems to have been widely accepted. It is found, for example, in Jean Bernoulli (1667-1748):

> To ensure that the time taken for the body to descend or ascend through any total arc is always the same, the value of $\int \frac{dt}{2g}$, which we have just found, must be equal to some similar function of zero dimension.

where g is the gravitational acceleration.[38] The same applies to Alexis Fontaine (1704-1771), who equates a zero-dimensional quantity with a numerical quantity:

> $F\,L\left(\frac{n^{-1/2}x}{\sqrt{X^2-x^2}}\right)$ is a zero-dimensional function of X and x; so if we had this Fluent, & substituted for x, X, we'd have T equal to a zero-dimensional function of X, i.e., a constant number.[39]

Occasionally, we find more advanced discussions. In 1761, François Daviet de Foncenex (1734 - 1799) published a memoir on the fundamental laws of mechanics, in which he used dimensional reasoning to determine the form of the resultant force of two other forces. He imagines two forces of equal magnitude F exerted on a body, subtending an angle A. Clearly, the resultant R is a function of F and A only. But since the angles are zero-dimensional, R and F must have the same dimensions. It follows that the resultant must be proportional to F. Foncenex points out that this method can be used to derive other

[38] Bernoulli 1730: 82 (art: 78-101). g is defined as "the gravity that animates bodies," Bernoulli 1730: 80.
[39] Fontaine 1734: 372 (art 371-379).

propositions.[40] This reasoning can be seen as an anticipation of Rayleigh's method, which we'll discuss in Chapter 7.

Martins sees Foncenex's work as the true emergence of dimensional analysis. In this respect, he notes:

> It is common in the history of science to find an idea used obscurely before it is explicitly defined. As we have shown, this was the case in dimensional analysis.[41]

It's true that Foncenex's reasoning can be seen in retrospect as an example of the application of dimensional analysis. Nevertheless, as Martins himself says, it remains obscure. And it's hard to imagine dimensional analysis in the modern sense without a precise designation of the quantities taken as fundamental. It's unlikely that Foncenex was aware of this need.

In 1765, Leonhard Euler (1707-1783) presented a discussion of the problem of homogeneity in his *Theoria motus corporum solidorum seu rigidorum*. In it, he expresses concern for the question of units, and presents dimensions as dependent on the choice of units.[42] His work in dynamics has been highly influential. In this context, if we translate his conceptions into modern dimensional terms, we can say that he recognizes only two fundamental units: length and force. As for time, its dimensions can be deduced from those of speed. For Euler, the measurement of speed is independent of that of time: he measures speed in terms of distance; it corresponds to the square root of the height an object must fall to reach said speed (the object being released at rest, of course). So $[v] = \mathrm{L}^{1/2}$. Since

[40] Foncenex 1761: 299-322.

[41] "But it is a common phenomenon in the history of science that an obscure idea is used before being explicitly defined. As has been shown, this was the case in dimensional analysis," Martins 1981: 336.

[42] It is possible that this concern for units was inspired by his formulation of the equation for the motion of a mass A due to a force p : $A\,dv = n\,p\,dx$

or, to use modern symbols, i.e. m for mass and F for force: $m\,dv = n\,F\,dx$

In this formula, Euler specifies that "n" is a constant whose value depends on the units chosen. (Euler 1736). Nevertheless, in this case n is dimensionless, and so it's unlikely that his concern with dimensions stems from this.

$[T] = \frac{L}{[v]}$, this implies for time: $[T] = L^{1/2}$. This also coincides with Euler's conceptions of mass in Newton's Second Law. As we can see, acceleration has no dimension. But Euler attributes to mass the units of a weight, i.e. a force.[43] Generally speaking, as Macagno points out, Euler believed that the dimensions of physical quantities are not absolute.[44]

Although in the 18th century the development of an algebraic physics in which quantities were treated as numerical - following the tradition represented above by de Muris - seemed to avoid the question of homogeneity, the dimensional coherence found in Euler in particular, although implicit, suggests that concern for homogeneity considerations guided his choice of natural units. In the 19th century, the issue of homogeneity reappeared explicitly and, with Fourier, took on a new form.[45]

1. 4 - The early 19th century: The resurgence of dimensions

Lazare Carnot (1753-1823) was again preoccupied with dimensions in 1803.[46] His approach to the question is surprising for two reasons. Firstly, he makes no reference to considerations of homogeneity. In fact, he doesn't even use the term dimension. And yet, Carnot offers a list of what would later be referred to as dimension formulas - hence the relevance of his work. These are not simply dimensions in relation to the length he considers, but in relation to two other quantities as well: mass and time - in fact, he doesn't

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[43] Mikhaïlov & Sedov 2007: 170-172.
[44] Macagno 1971: 393.
[45] For the impact of algebraic physics on dimensional considerations in the second half of the 18th century, see Roche 1998: 190.
[46] Carnot 1803.

even mention length first. Carnot thus expresses the most common physical quantities in mechanics in terms of these three quantities. He begins by defining these quantities: [47]

> 17. A space covered, or any line divided by a time, is generally called a *speed*.
> A mass multiplied by a velocity, or the product of a mass and a line divided by a time, is called *momentum*.
> A speed divided by a time is called an *accelerating* or *retarding force*. So an accelerating or retarding force is, in general, the quotient of a space or line divided by the square of a time.[48]

and so on for the other quantities.[49] The need to reduce physical quantities to mass, time and space is already present in the definitions themselves.

Carnot then presents the part of his work that interests us:

> 18. So, in general, if *m* represents a mass, *e* a space, or linear quantity, *t* a time,
> Any quantity of the form, or reducible to the form $\frac{e}{t}$ is called speed.
> Any quantity of this form $m\frac{e}{t}$ is called momentum.
> Any quantity of this form $\frac{e}{t^2}$ is called an accelerating or retarding force.
> Any quantity of this form $m\frac{e}{t^2}$ is called motive force.
> Any quantity of this form $m\frac{e}{t}$, or of this one $m\frac{e}{t^2}$, is simply called force or power.
> Any quantity of this form $m\frac{e^2}{t^2}$ is called living force, moment of motive force, or moment of activity.
> Finally, any quantity of this form $m\frac{e^2}{t}$ is called momentum, or action quantity.[50]

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[47] Carnot 1803: 12.
[48] Carnot 1803: 11-12.
[49] "The product of a mass and an accelerating or retarding force is generally called the *driving force*.
The simple term "*force*" or "*power*" covers both quantities of motion and driving forces.
The product of a mass and the square of a velocity, or the product of two velocities, or, what amounts to the same thing, the product of a line and a motive force, is called *living force*, *motive force moment, activity moment*.
The product of a mass by a velocity and by a line, or by the square of a velocity, and by a time, is called *moment of momentum*, or *quantity of action*."
Carnot 1803: 12.
[50] Carnot 1803: 12-13.

It seems, then, that Carnot saw in these formulas a way of defining classes of quantities in terms of mass, space and time. Despite their resemblance to dimension formulas as they would be conceived in the second half of the 19th century, there is nothing here to suggest a desire to develop criteria for ensuring the homogeneity of equations. Nevertheless, it is not inconceivable that Carnot's work may have inspired later attempts in this direction. Indeed, in the footnote to the first definition cited (that of velocity), we find the following remarks:

> These two quantities, space and time, being heterogeneous, it is not precisely the quotient of one by the other that is at issue, but the quotient of the ratios that these quantities have to their respective units, in accordance with the accepted usage in geometry, when it comes to dividing, for example, a surface by a line, or multiplying one line by another. This is also the case in all parts of mathematics, when it comes to multiplying concrete quantities by one another, or dividing heterogeneous quantities by one another. It is not these quantities themselves that are expressed algebraically, but the abstract numbers that form the quotients of these quantities by their respective units. [51]

Carnot thus justifies his way of expressing himself by the fact that the equations to which his definitions refer are understood as numerical equations - a remark which, as such, could not be more innocent, simply reflecting the conception that had become customary at the time. Carnot was far from saying that two quantities are mutually homogeneous when they can be expressed by the same formula as a function of e, t and m. Nevertheless, his remark draws attention to the heterogeneous nature of the three quantities involved in his formulas in section 18, and raises the question of homogeneity in general.

In addition, Carnot's note draws attention to an aspect of the question that is soon to assume great importance: units. In order to emphasize the numerical character of the quantities concerned, he explains that these are not the physical quantities themselves, but

[51] Carnot 1803: 11.

the "relations which these quantities have to their respective units." As we'll see in the next section, the notion of unit - and even, more precisely, of a ratio involving the unit of a quantity - became essential to Fourier's definition of dimension in the 1820s.

We also find an interesting use of homogeneity in 1816, in Siméon Denis Poisson (1781-1840).[52] Poisson wrote after Fourier's first work on homogeneity (but before his main discussion), and may well have been inspired by it. This is not certain, however, and unlike Fourier, Poisson does not use the term "dimension" in a sense that interests us.

The problem of interest to Poisson was the one presented by the Académie des Sciences for its 1816 prize: we consider an infinitely deep, weightable fluid, initially at rest, which is set in motion by a given action; the aim is to determine the shape of the liquid surface and the movement of its molecules after a given time. In this context, Poisson challenged the results proposed by Lagrange in 1782. According to Lagrange, waves propagate on the surface of a liquid with a velocity that is both constant and independent of the shaking that gives rise to them. Poisson believes this to be true only in the case of a shallow fluid, but cannot be extended to the general case as Lagrange does.[53] He uses the principle of homogeneity to demonstrate this.

Poisson imagines that the wave is due to an object initially immersed in the fluid, which is then at rest, and which is withdrawn vertically. His reasoning is based on the idea that, since the depth of the fluid can no longer be taken into account, "the only lines that are included among the data of the question, are the dimensions of the immersed body, and the space that a weighing body travels in a determined time."[54] Although Poisson does not

[52] Poisson 1815: 71-186. See Darrigol 2005: 37-38.
[53] Poisson 1815: 73-75.
[54] Poisson 1815: 74.

use the term "dimension" in this sense, it is clear that what he means by "lines" covers the same concept as that of "dimension" traditionally employed in the context of the homogeneity principle. He wishes to establish a relationship between the distance travelled by the wave in a given time, and these "lines."

Poisson considers two scenarios. In the first, he assumes that the velocity of the wave is independent of the initial shaking, and therefore does not depend on the dimensions of the body used to create it. The principle of homogeneity then implies that the distance traveled by the wave must be proportional to that which a falling body would travel under the effect of terrestrial gravity: "The motion of waves will be similar to that of grave bodies, with an acceleration which will be a certain multiple, or a certain fraction of the acceleration of gravity."[55] The speed of waves is therefore not constant.

In the second case, Poisson wants to obtain a uniform velocity. To achieve this, the distance covered by the waves must also depend on one of the dimensions of the immersed body - let's call it *l*, for example. To satisfy the principle of homogeneity, both dimensions must now intervene to the power of ½, which Poisson obtains by taking their geometric mean. The distance travelled by a body in free fall being $\frac{1}{2} g t^2$, the distance travelled by the wave must therefore be proportional to $\sqrt{g t^2 l}$ (which Poisson does not explicitly give). This corresponds to a uniform velocity, but we have to assume that the wave's propagation speed depends on the initial shaking. Poisson even considers an intermediate case:

> It could also happen that the movement of the waves is accelerated, and that the acceleration depends on the numerical relationship between these dimensions: it's up to the calculator to decide which of these movements should actually take place.[56]

[55] Poisson 1815: 75.
[56] Poisson 1815: 75.

What interests us here is how Poisson arrives at his conclusions using the homogeneity principle. He uses no symbols in his demonstrations. It's clear that he reasons only in terms of geometric dimensions and distances. He does not appear to apply homogeneity to time, even though he equates the expression we have given, $\sqrt{g\,t^2 l}$, with speed.

Poisson's originality therefore consists in obtaining a relationship between the speed of a wave and other physical quantities by employing the principle of homogeneity over lengths, to a fractional power.

1. 4 - Fourier's contributions

From the second half of th e 19th century onwards, all the sources that mention the subject agree in naming Fourier as the originator of dimensional analysis. They refer to his discussion in *Théorie analytique de la chaleur*, published in 1822. However, antecedents to these ideas can be found in Fourier's earlier work, his *Théorie du mouvement de la chaleur dans les corps solides* (submitted in 1811, and presented to the Académie des Sciences in 1812).[57]

As their titles indicate, both of Fourier's works are concerned with the study of heat phenomena. The study of heat in the 18th century had mainly been carried out by chemists and physicians. Several conceptions clashed as to the nature of heat and temperature. In the tradition represented by Antoine Laurent Lavoisier (1743-1794), heat is a substance, caloric, which can be found in two different states, free or combined:

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[57] Théorie du mouvement de la chaleur dans les corps solides, 1ère Partie, *Mémoires de l'Académie royale des sciences de l'Institut de France*, 4 (1819-20, publ. 1824), 185-555; 5 (1821-22, publ. 1826) 153-246.

> Free caloric is that which is not involved in any combination. As we live in the midst of a system of bodies with which caloric has adhesion, it follows that we never obtain this principle in the state of absolute freedom.
> Combined caloric is that which is bound in bodies by the force of affinity or attraction, & which constitutes part of their substance, even their solidity.
> By this expression specific caloric of bodies, we mean the quantity of caloric respectively necessary to raise by the same number of degrees the temperature of several bodies equal in weight. This quantity of caloric depends on the distance between the molecules of the bodies, on their greater or lesser adherence; & it is this distance, or rather the space that results from it, that we have named, as I have already observed, the capacity to contain the caloric.[58]

In this tradition, temperature is considered to be the density of free caloric only. Another school was that of the Scottish physician William Irvine (1743-1787) and his followers, who defined the temperature of a body as the amount of heat contained in it divided by its calorific capacity. Kinetic conceptions of heat did exist, but they were very much in the minority, and the best-known of them, that of Rumford (born Benjamin Thompson, 1753-1814), differed from its modern colleague in that he conceived of molecules as animated by vibratory motion, not translation. Among the most enduring contributions of the time came in 1782 from the Scotsman Joseph Black, who developed the concepts of latent heat and specific heat in his *Lectures on Chemistry*. These concepts were based on the idea of heat conservation. The famous 1783 memoir by Pierre-Simon Laplace (1749-1827) and Lavoisier makes reference to them. The various modes of heat exchange were also defined in the second half of the 18th century: in 1777, Karl Scheele defined radiant heat, and in the 1790s, conduction, by Jan Ingen-Housz (1730-1799) and Rumford, who also demonstrated the occurrence of convection. This perception of heat in dynamic rather than static terms attracted the attention of physicists and geometers: the great Laplace we've already mentioned, his disciple Jean-Baptiste Biot, as well as a British mathematician and

[58] Lavoisier 1789: 21.

physicist, John Leslie, whose work *An experimental enquiry* (1804) was a great success not only in Great Britain but also in France, and finally, of course, Fourier in the decades that followed. [59]

From a philosophical point of view, Fourier's positions are distinguished by an economy of hypotheses. His aim is not to establish the nature of heat, or to explain its characteristics through models involving more fundamental mechanisms. For Fourier, a phenomenon such as heat is self-sufficient: it serves as a basis for the explanation of more complex phenomena, but does not require deeper explanations, which would necessarily be hypothetical in nature. Auguste Comte (1798-1857) later saw Fourier's work on the diffusion of heat as a paradigmatic example of the positivist approach he advocated.[60] Comte even dedicated the whole of his illustrious *Cours de philosophie positive* to Fourier (and Blainville).[61]

The first text in which Fourier shows an interest in dimensions is a dissertation presented in 1811[62] and published in 1819-1820.[63] The relevant part is as follows:

> In calculations, the numerical values h and K for external conductivity and specific conductivity are referred to by the numbers h and K, but the same letters h and K are sometimes used to designate the ratios $\frac{h}{\mathrm{K}}$ and $\frac{\mathrm{K}}{\mathrm{C\,D}}$.
> This double use may have led to a few transcription or calculation errors: these can easily be rectified, bearing in mind the homogeneity of the terms. To do this, simply note that the quantities
> *x,* number of length units, the dimension...1

[59] Cardwell 1977: 138-140 [138-145]; Chang 2004: chapter 4, 159-219.
[60] Grattan-Guinness 1972: 477-478.
[61] Comte 1830-1842, vol 1 (1830).
[62] The title page of the 1819 publication notes: "Ce Mémoire est la copie littérale de la pièce déposée aux archives de l'Institut le 28 Septembre 1811, et qui a été couronnée dans la séance publique du 6 janvier 1812. *No changes have been made to the text of this piece*."
[63] Fourier 1819-1820: 185-525; 455-456.

> *h,* external conductivity, dimension..-2
> K*,* inherent conductivity, dimension...-1
> D, density...-3
> C, specific heat capacity...0
> Z, temperature...0
> and *t*, elapsed time, the dimension...0
>
> This number of dimensions is that which results from the definitions given for the quantities *h*, K, C, etc. When we take into account the number of dimensions of each letter, we'll find that all equations are composed of homogeneous terms, and it will be easy to recognize which quantities are designated by *h* and K.[64]

We can see that, unlike Carnot, Fourier does use the term dimension, and that he is motivated by a concern to verify equations by using the homogeneity condition. But he uses the notion of dimension only in the spatial sense. This could be attributed to his preoccupation with calorific phenomena in which time and mass are not directly involved. Nevertheless, from a theoretical point of view, this work is hardly original compared to those that preceded it, and in terms of the quantities taken into account, it in no way prefigures what we'll find in the *Théorie analytique de la chaleur*. The notion of a negative dimension is a given, since it dates back to Wallis.[65] It is truly indispensable for achieving homogeneity in this way.

It's surprising that Fourier doesn't go into more detail about his method. He doesn't specify how he obtains the dimensions of the quantities in question. Nor does he explain how the principle of homogeneity is to be applied. He indicates only that the dimensions of a quantity are obtained from the equation defining it, and that the principle of homogeneity must be applied to each of the terms of an equation. He certainly seems to suggest that the need to respect this principle also makes it possible to identify certain

[64] Fourier 1819-1820: 456.
[65] Wallis 1695: 103.

quantities, enabling us to determine whether it's self-conductivity or external conductivity we're dealing with. But once again, Fourier doesn't explain how to proceed.

Should we deduce from such brevity on the subject that Fourier expected his readers to have a certain familiarity with it? This is unlikely. Even if the considerations Fourier evokes here go back to Wallis. There is no reason to suppose that the latter's conceptions, or those of the later scholars we have mentioned, were widely known. More likely, Fourier could reasonably expect his readers to understand his reasoning without being familiar with such considerations. Indeed, the term "dimension" already referred to a power exponent - and had done so for a long time. For example, in Montferrier's *Dictionnaire des sciences mathématiques pures et appliquées*, published in 1845, we find the following two definitions:

> DIMENSION (Geom.). Length, width or thickness of a body. We conceive of *lines* as having only one dimension, *length*; *surfaces* as having only two dimensions, *length* and *width*, and *solids* as having three dimensions *length*, *width* and *thickness* or *depth*. *See*. Line, Solid, Surface.
> The word dimension is still used in algebra to designate the degree of a power or an equation [...] In general, a quantity has as many dimensions as there are factors in its composition: *a*, for example, is of one dimension, *ab* is of two, *abc* of three, *abcd* of four, and so on.

Neither definition corresponds to Fourier's definition of "dimension". Nevertheless, Fourier does refer to powers of lengths. He probably felt that enumerating the results was sufficient to avoid any ambiguity regarding the rules of calculation to be applied, namely that when two quantities are multiplied by each other in one of the terms of an equation, their dimensions should be added rather than multiplied - as with any exponent - and that when dealing with a ratio of two quantities, they should be subtracted. What's more, once the reader has convinced himself of this rule by applying it to the equations that define the various quantities listed by Fourier and comparing his own results with those given by the author, it goes without saying that the same rule also applies to the terms of the equation we're trying to verify by applying the principle of homogeneity.

It was in his famous treatise *Théorie analytique de la chaleur*, published in 1822, that Fourier truly developed the concept of dimensions beyond its previous meaning, and laid the foundations for dimensional analysis.[66] This work introduces two important innovations. Firstly, the concept of dimension is defined on the basis of considerations of unit change. Secondly, it now applies to quantities other than geometric ones. As we have seen, Descartes and Wallis had already used the term in this extended sense, but this practice was not followed after them. [67]

Fourier said he had several motivations for tackling the problem of homogeneity and dimensions. Some can be described as practical:

> Each indeterminate or constant quantity has a dimension of its own, and...the terms of the same equation could not be compared, if they did not have the same *dimension exponent*.[68] We have introduced this consideration into the theory of heat to make our definitions more fixed, and to serve as a check on the calculation.[69]

Ensuring the homogeneity of equations seems to be an important motivation for Fourier, as he returns to it at the end of his discussion of the subject,[70] even illustrating the point with a specific example. Moreover, the way Fourier determines dimension exponents

[66] Fourier 1822: 154-158.
[67] While it's conceivable that Fourier was aware of their thoughts on the subject, there's no real indication of this. In any case, Fourier makes no mention of them.
[68] Italics in the original.
[69] Fourier 1822: 154-155.
[70] "By applying the preceding rule to the various equations and their transforms, we will find that they are homogeneous with respect to each kind of unit, [...]. If this were not the case, some error would have been made in the calculation, or abbreviated expressions would have been introduced." 158.

invites the use of the concept for conversion between systems of units. Fourier himself doesn't *explicitly* suggest this application, but it's almost self-evident.[71]

Fourier didn't introduce the question of dimensions for purely practical reasons. On the contrary, it is of fundamental theoretical importance. He states:

> It [the consideration of homogeneity] derives from primordial notions about quantities; for this reason, in geometry and mechanics, it is equivalent to the fundamental lemmas that the Greeks left us without demonstrations.[72]

Which lemmas Fourier is referring to, and what exactly this fundamental character of dimensional calculus consists of, is unfortunately less clear. Martins sees this as a sign that Fourier conceived the principles in question as a generalization of geometric homogeneity, and that he had been inspired in this by the work of Foncenex:

> This sentence becomes clearer if we assume that he was familiar with the work of Foncenex and Legendre. These authors had shown that the principle of dimensional homogeneity made it possible to demonstrate some fundamental laws that had not been demonstrated by the Greeks: the law of the composition of forces, and the fifth postulate of Euclid's geometry.[73]

Indeed, Fourier had read Foncenex. But given the absence of the concept of force in antiquity, it's hard to see the connection. What's more, Foncenex used the concept of dimensions to demonstrate propositions (such as that the resultant of two forces of the same magnitude is proportional to that force), which Fourier did not. It would seem more likely that Fourier was motivated by considerations of invariance, from which he derived his interest in the problem of conversion between systems of units. This is all the more likely

[71] Fourier 1822: section 161, 155-156.
[72] Fourier 1822: 155.
[73] "It is difficult to explain this sentence if we suppose that Fourier did not know the previous use of the concept of dimension in Geometry and Mechanics. But the sentence becomes clear if we assume that he knew the works of Foncenex and Legendre. These authors showed that the principle of dimensional homogeneity allowed the demonstration of some basic laws that were not demonstrated by the Greeks: the law of composition of forces, and the fifth postulate ofEuclid 's geometry." Martins 1981: 335.

given that, if Fourier saw the latter problem solely in terms of practical considerations, we might expect him to give a concrete example of the use of dimensions for conversion purposes, which he does not. It would seem that, in fact, he focuses on the problem of conversion because it enables us to distinguish between, on the one hand, what is arbitrary and contingent, and, on the other, what is necessary and fundamental in an equation: conversion is, in essence, the operation that affects the former elements and not the latter. And what is fundamental, absolute, what expresses the very nature of an equation, is the relationship it expresses between the quantities to which it applies:

> In the analytical theory of heat, every equation (E) expresses a necessary relationship between subsistent quantities x, t, v, c, h, k. This relationship does not depend on the choice of the unit of length, which by its very nature is contingent, i.e., if we took a different unit to measure linear dimensions, equation (E) would still be the same.[74]

And further on:

> Equation (E) does not need to undergo any change, if we write, instead of x, mx, and at the same time ,$\frac{k}{m}\ \frac{h}{m^2}\ \frac{c}{m^3}$, instead of k, h, c; the number m will disappear on its own after these substitutions.[75]

Fourier introduces the question of dimensions by taking as an example the most natural physical quantity when speaking of measurement, which is also the one to which the term had previously referred: length. As the passage quoted above indicates, he imagines an equation in which this quantity is involved, and in which it takes x as its numerical value, in the original system of units. He then assumes that the unit of length is

[74] Fourier 1822: 155.
[75] Fourier 1822: 156.

changed in such a way that the second unit of length is equal to the first divided by a number m :

Unit of length in unit system n°1 : Unit (1)
Unit of length in the n°2 unit system: Unit (2) = Unit (1) / m

The numerical value representing this physical quantity (i.e. a length which he designates by $\overline{\mathrm{ab}}$) in the second system of units, is then equal to the numerical value of this quantity in the first system of units multiplied by m :

Numerical value of the quantity (length "$\overline{\mathrm{ab}}$ ") in the n°1 unit system: x
Numerical value of the quantity (length "$\overline{\mathrm{ab}}$ ") in the n°2 unit system: $x\,m$

or $.\, x\, m^1$

This would give us, for example :

unit (1): meter, unit (2): centimeter = meter / 100
numerical value of "$\overline{\mathrm{ab}}$ " in terms of unit (1) (i.e. m): 2
numerical value of "$\overline{\mathrm{ab}}$ " in terms of unit (2) (i.e. cm): 2×100

What Fourier calls "dimension" - or, more precisely, "dimension exponent"[76] - is none other than the exponent of *m*, i.e. the exponent of the factor by which the numerical value of a quantity must be multiplied in order to pass from one system of units to another.[77]

However, for the equation to remain valid, the other quantities must also be multiplied by a factor containing *m*. In Fourier's case, these other quantities are what he

[76] Fourier seems to use the term "dimension" as a kind of shorthand for the full expression "dimension exponent." For example, when he gives the table we report below (Table 1), he states "the following table represents the dimensions of the three indeterminates and the three constants" and in the table itself he entitles the entries in question "dimension exponent of."
[77] Fourier 1822: 156.

calls "specific elements", i.e. coefficients representing physical properties specific to a given physical system. Given their importance here, we need to define them precisely.

These "specific elements" are three in number: "intrinsic conductivity" (*k* or *K*), "conductivity relative to atmospheric air" (*h)* and "specific heat capacity" (*c)*. The first, *k*, corresponds to what we would call thermal conductivity, and *c* to specific heat. More precisely, Fourier defines *k* as follows: [78]

> The measure of the specific conductivity of a given substance is the quantity of heat which, in an infinite solid formed from this substance and enclosed between two parallel planes, flows during the unit of time through a surface equal to the unit, and taken on any intermediate plane, parallel to the outer planes whose distance is equal to the unit of measurement, and one of which is maintained at temperature 1, and the other at temperature 0. The coefficient K is the constant flow of heat through the entire prism, and is the measure of conductivity.[79]

This can be represented as follows:

Interior of an infinite solid

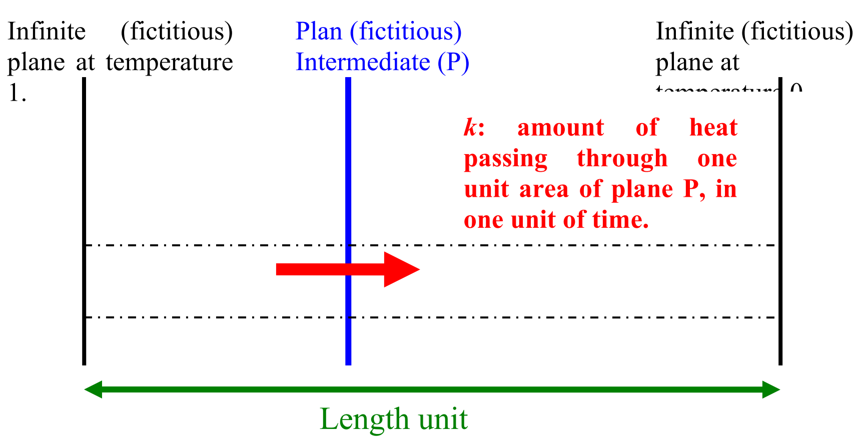

Figure 1.1: definition of *k*

As its name suggests, *h is* intended to express the heat transfer between a substance and the surrounding air. It is defined as the numerical value representing "the quantity of heat

[78] Fourier uses capital K in the quotation given here, but this definition does indeed refer to the quantity *k* we are concerned with, as Fourier himself refers to this definition (i.e. to article 135).
[79] Fourier 1822: 127.

that exits during the unit time, from the unit surface at temperature 1,"[80] or, more precisely

:

> *[h]* expresses the quantity of heat which would pass, during a given time (one minute), from the surface of this body into the atmospheric air, assuming that the surface has a given extent (one square meter), that the constant temperature of the body is 1, that that of the air is 0, and that the heated surface is exposed to an air current of a given variable speed. The quantity of heat expressed by the coefficient *[h]* is made up of two distinct parts... One is the heat communicated by contact with the surrounding air; the other, much less than the first, is the radiant heat emitted.[81]

This can be illustrated as follows:

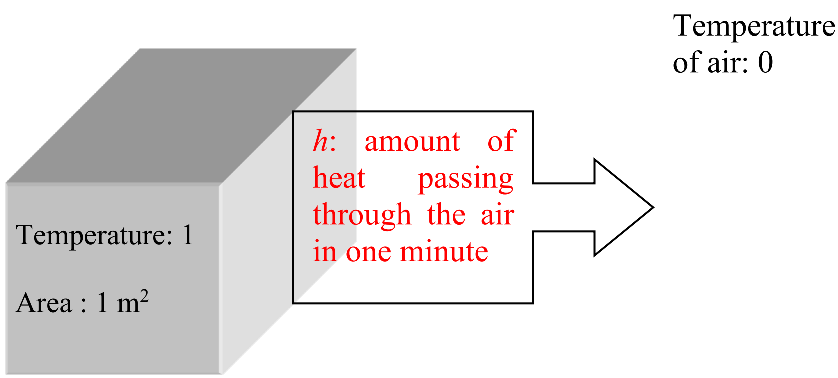

Figure 1.2: definition of *h*

As for the coefficient *c* (the specific heat capacity), Fourier first defined it in relation to a given weight of matter: "[it] expresses what heat is required to raise the unit weight of the substance in question from temperature 0 to temperature 1;"[82] This corresponds to the most common definition at the time, as Fourier himself says.[83] Nevertheless, when he comes to deal with the problem of dimensions, he redefines it as "the quantity [of heat] necessary to raise from temperature 0 to temperature1 1 'unit of

[80] Fourier 1822: 155.
[81] Fourier 1822: 25.
[82] Fourier 1822: 102.
[83] Fourier 1822: 154.

volume of a given substance, and not the unit of weight of that substance."[84] The reason for this is that the other quantities in his equations don't involve weight, so he can dispense with it altogether: Fourier doesn't consider the dimension of the quantities he's interested in in relation to weight.[85]

The three quantities *k*, *h* and *c* are characteristic of the substance of which the body in question is composed. It is true that *h* also depends on "the more or less polished state of the surface;...."[86] So, rather than being characteristic of a given substance, it is characteristic of a given system. Nevertheless, Fourier seems to overlook this effect: "[*c*, *h*, and *k*] are, for each substance, constant elements that experience makes know." [87]

Let's see how Fourier's principle of homogeneity applies to these coefficients. Fourier, it should be remembered, simply states that for an equation involving them :

> Equation (E) should not undergo any change, if we write, instead of x, *mx*, and at the same time , $\frac{k}{m}$ $\frac{h}{m^2}$, and $\frac{c}{m^3}$ instead of *k*, *h*, *c*; the number *m* will disappear by itself after these substitutions. [88]

where *m* is the conversion factor between two units of length (and *x* the numerical value of a given length expressed in terms of the first unit).[89]

Fourier leaves it to his readers to understand why. This seems obvious in the case of *c* and *h*. Indeed, *c* refers to a given unit of volume. As we saw above, writing *mx* instead

[84] Fourier 1822: 154.
[85] Fourier 1822: 154 & table 157.
[86] Fourier 1822: 8-9.
[87] Fourier 1822: 154.
[88] Fourier 1822: 156.
[89] Fourier 1822: 154.

of x is equivalent to dividing the unit of length by m. This unit therefore changes from u_1 to $u_2 = \frac{u_1}{m}$. Consequently, the unit of volume is reduced by a factor of $\frac{1}{m^3}$, from $u_1^{\,3}$ to $u_2^{\,3} = \frac{u_1^{\,3}}{m^3}$. The capacity, being the quantity of heat contained in this volume, must also be reduced - i.e. we must now take $\frac{c}{m^3}$ instead of c.

The coefficient h refers to a unit area, being the amount of heat escaping from a (unit) area. When the unit length changes from u_1 to $u_2 = \frac{u_1}{m}$, the surface area is reduced by a factor of $\frac{1}{m^2}$, and so is this quantity of heat.

Note that this reasoning requires us to recognize that the dimension of a surface is the square of a length, and that a volume is the cube of a length. As we have seen, this idea had been accepted for centuries. Nevertheless, the fact that it is used here shows that Fourier's intention was to extend the concept of dimension, giving it a broader meaning that encompassed the one it had previously had: as far as geometric quantities are concerned, "dimension" means the same thing as before.

The situation is slightly more complex for the coefficient k, which represents the conductivity inside the body. k is defined as a quantity of heat passing through a surface, just like h, and we would therefore expect it to also decrease by a factor $\frac{1}{m^2}$. However, if we go from u_1 to $u_2 = \frac{u_1}{m}$, we also decrease the distance between the two infinite planes (see Fig.1). Assuming that the speed of heat propagation is unaffected, a given quantity of heat that leaves the plane at temperature 1 now arrives at the plane at temperature 0 in m

times less time. Now, k is the amount of heat passing through one unit area of the P plane, *in one unit of time*. As a result, the heat flow through P is increased by a factor of m. We therefore need to reduce k by $.\frac{1}{m^2}m = \frac{1}{m}$

Fourier is also careful to specify which quantities should not be modified when changing units, namely angles and trigonometric functions, as well as logarithms and power exponents, which he calls "*absolute* numbers" and "abstract numbers".[90] In this case, the dimension to be assigned to them is zero.

The modifications described by Fourier so far concern a change in the unit of *length*. It should be stressed, however, that the originality of Fourier's work in the *Théorie de la chaleur* lies in his belief that similar modifications must be made when other units are changed. Fourier identifies three other physical quantities that, like length, require such a procedure: time, temperature and weight.[91] As John Roche notes, the quantity of heat was no doubt part of this list in Fourier's mind, bringing the number of basic quantities to five. Indeed, Fourier says above:

> To measure [the quantities we analyze] and express them in numbers, we compare them to various kinds of units, of which there are five, namely: the unit of length, the unit of time, the unit of temperature, the unit of weight, and finally the unit used to measure quantities of heat.[92]

Each of these three quantities (time, temperature and weight) has a corresponding dimensional exponent for the quantities involved in an equation. We'll call these three quantities "fundamental quantities" or "fundamental quantities". [93]

[90] Fourier 1822: 156; italics in original.
[91] Fourier 1822: 156-157.
[92] Fourier 1822: 153.
[93] The expression that will quickly come into use is "fundamental unit", but we prefer to avoid it here, as the term "unit" suggests that they are specific to a particular system of units, which is obviously not the case.

Fourier explicitly indicates the modifications to be made to the physical quantities and specific coefficients involved in an equation when the unit of time or temperature changes, as well as the resulting dimensions.[94] This leads him to draw up the following table:

Exhibitor size of...	**LENGTH**	**DURATION**	**TEMPERATURE**
x	1	0	0
t	0	1	0
v (temperature)	0	0	1
specific *conductivity.................k*	-1	-1	-1
surface *conductivity……….......h*	-2	-1	-1
heat *capacity.............................c*	-3	0	-1

Table 1.1[95]

Each fundamental quantity has dimension 1 with respect to that same quantity and zero with respect to the others. Once again, this property is defined in terms of the changes to be made to quantities when converting between systems of units: length has dimension zero with respect to time, because changing the unit of time does not affect the numerical value that expresses a given length. [96]

To compensate for the brevity of his explanations, at the end of his discussion Fourier verifies the homogeneity of a precise equation "with respect to each kind of unit":
[97]

[94] He is not explicit about the weight, which he does not indicate in his table, but the reason for this is that he has ensured that this quantity does not play a part in his equations.
[95] Fourier 1822: 157.
[96] Fourier 1822: 156.
[97] Fourier 1822: 157-158.

$$\frac{dv}{dt} = \frac{K}{C.D} . \frac{d'v}{dx^2} - \frac{hl}{C.D.S} v \quad [98] \qquad \textbf{(3)}$$

where v is temperature, l is distance and S is area. This equation represents the propagation of heat within a ring of rectangular cross-section: [99]

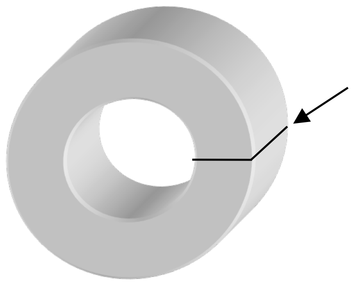

Figure 1.3

Fourier doesn't bother to specify how to determine the dimensions of the differential quantities, but it's obvious that he treats them as quotients: $\frac{dv}{dt}$ has the same dimensions as $\frac{v}{t}$. He simply indicates the dimension of the three terms with respect to each of the fundamental quantities - length, temperature and time, leaving it to his readers to determine why. These terms must have dimensions 0 with respect to the unit of length, 1 with respect to the unit of temperature, and -1 with respect to the unit of time. The calculations leading to this result can be summarized in the following table, where l represents the unit of length, v the unit of temperature, and t the unit of time:

First term **:** $\frac{dv}{dt}$	**Second term:**	Third term: $\frac{hl}{C.D.S}v$

[98] In reality, this complete equation is given 102 (section 105) of the *Analytical Theory of Heat*. The equation given in the part of the text under discussion here (*Analytical Theory of Heat*, 158, i.e. section 162) is as follows:
$\frac{dv}{dt} = \frac{K}{C.D} . \frac{d'v}{dx^2} - \frac{hl}{C.D.S}$ but the fact that the "v" factor no longer appears in the second term on the right is clearly an oversight - and without it, the equation is no longer homogeneous with respect to temperature.
[99] Fourier 1822: Fourier derives it in sections 104 and 105, pp.101-103, and describes the ring or armille on p.100.

				$\frac{K}{C.D}.\frac{d'v}{dx^2}$							
	Size dimensions				Size dimensions				Size dimensions		
	l	***v***	***t***		***l***	***V***	***t***		***l***	***v***	***t***
v	0	1	0	K	-1	-1	-1	h	-2	-1	-1
t	0	0	1	$C.D = c$	-3	-1	0	l	1	0	0
				v	0	1	0	$C.D = c$	-3	-1	0
				x	1	0	0	S	2	0	0
								v	0	1	0
Calculations	line 1 - line 2			Calculations	line 1 - line 2 + line 3 - (2 x line 4)			Calculations	line 1+ line 2 - line 3 - line 4 + line 5		
Term Dimensions	**0**	**1**	**-1**	Term Dimensions	**0**	**1**	**-1**	Term Dimensions	**0**	**1**	**-1**

Table 1.2

Fourier briefly gives a second example, with the aim of showing that an exponent always has 0 dimension, whatever the unit with respect to which its dimension is considered. The example in question is taken from an equation defined in Article 76 :

$$v = A\mathrm{e}^{-x\sqrt{\frac{2h}{kl}}} \tag{4}$$

All dimensions of the exponent, $x\sqrt{\frac{2h}{kl}}$ are 0, which is easily verified:

$$x\sqrt{\frac{2h}{kl}}$$

	Size dimensions		
	l	***V***	***t***
x	1	0	0
h	-2	-1	-1
k	-1	-1	-1
l	1	0	0
Calculations	line 1 + ½ line 2 - ½ (line 3 + line 4)		
Term dimensions	**0**	**0**	**0**

Table 1.3

Nevertheless, it's surprising that Fourier doesn't explain why the dimension of an exponent, or more generally that of the argument of many functions (logarithm, sine, etc.), must be 0. No doubt this was obvious to him.

A first question concerns the absolute or relative nature of the dimensions of physical quantities: to what extent are these dimensions characteristic of the quantities to which they relate? In particular, according to Fourier's description, can the same set of dimensions (e.g., 1 for length, 1 for time, 0 for temperature) correspond to several different physical quantities? Fourier doesn't broach the subject, but there's nothing in his discussion to suggest that he wouldn't think so.

Conversely, can a given quantity have different sets of dimensions? For a given set of fundamental quantities, it would seem that Fourier believes not - although, again, he doesn't explicitly discuss this point. On the other hand, the question of whether the dimensions attributed to a quantity depend on the quantities we have given ourselves as fundamental requires more attention.

Fourier chose to eliminate all reference to weight in his equations by redefining the coefficient *C* (specific heat capacity) in relation to weight, into its equivalent in relation to volume - which he called *c* - with the explicit aim of avoiding having to take into account dimensional exponents in relation to weight. This seems to imply that, according to him, a given physical quantity can have several sets of dimensional exponents characterizing it, each of these sets corresponding to a given group of fundamental quantities. In our view, however, such an interpretation is incorrect. For, in the case discussed by Fourier, the way in which he redefines *C* (so as to be able to express it in terms of a different set of fundamental quantities) shows that it is no longer actually the same quantity. Recall that in the first case discussed by Fourier, *C* was the capacity for a given *weight of* matter, while in the second case, *c* is the capacity for a given *volume*. Certainly, these two quantities embody the same phenomenon: heat capacity, i.e. the ability of a body to contain a greater or lesser quantity of heat. It's also true that Fourier, in defining *c*, presents it as differing from *C* by only one numerical factor:

> The number *C*, which enters into the calculation, is always multiplied by the density *D*, i.e., by the number of weight units equivalent to the weight of the volume unit; thus this product *CD* can be replaced by the coefficient *c.*[100]

But the said constant is in fact not a constant, since the density of a solid varies with its temperature.

Moreover, one cannot really speak of a change in the set of fundamental quantities when Fourier passes from *C to c*. What changes are the dimensional exponents with respect to the same fundamental quantities. [101]

[100] Fourier 1822: 154.
[101] Fourier 1822: 157.

capacity dimension exponent	Length	Duration	Temperature	Weight
defined in terms of weight (i.e. C)	0	0	-1	-1
defined in terms of volume (i.e. c)	-3	0	-1	0

Table 1.4

A directly related question is that of the criteria guiding Fourier's choice of fundamental quantities. Fourier does not address this subject explicitly either. However, since he introduces the concepts of dimension and dimension exponent by considering problems of conversion between systems of units, it seems reasonable that his choice of basic quantities should be partly based on such considerations; and indeed Fourier seems to take as basic quantities those quantities whose change of units clearly influences the value of other quantities.

The choice of Fourier may also have been motivated by the idea that its basic quantities are intuitively more "fundamental." In the case of space and time, such intuitive evidence ties in with the fact that space and time are the first entities defined by Newton in his *Principia*.[102] The same cannot be said of temperature. It seems, then, that Fourier chose his fundamental quantities not so much for theoretical reasons, but simply because they were the ones he had to measure in experiments carried out in the field he was interested in.

These remarks should be read in conjunction with what Fourier states at the very beginning of his discussion:

> In analytical heat theory, any equation (E) expresses a necessary relationship between subsistent quantities x, t, v, c, h, k.[103]

[102] Newton 1687, translated by A. Motte & F. Cajori 1934: 8 (scholium).
[103] Fourier 1822: 155.

Of these six quantities, the fundamental quantities are the three that do not represent specific coefficients. In keeping with his phenomenological credo, Fourier expresses his equations as far as possible in terms of measurable physical quantities.

1. 5 -Conclusions

As already mentioned, Fourier brings two profound innovations to the question of homogeneity. Firstly, he redefined the concept of dimension in the context of the question of changing units. From an epistemological point of view, this gesture is undoubtedly linked to his positivist attitude: ultimately, it is in reference to measurements that Fourier defines dimensions, via the problem of conversion between units of measurement. Secondly, he extends the concept of dimension to non-geometric quantities. These two aspects are not independent: defining dimensions in terms of conversion allows us to extend the concept to any measurable quantity, and to precisely define the exponents of dimension in all generality. Although Descartes and Wallis used the term dimension for non-geometric quantities, they did not clearly define the value of the associated exponents.

It's hard to imagine what problems or considerations might have led Fourier to these innovations - apart perhaps from Carnot's work, if he was aware of it. Paradoxically, it is conceivable that the very current that had led to the neglect of dimensional considerations in the previous century, namely, the development of a purely numerical conception of equations, was at the origin of these ideas. On the one hand, all quantities involved in an equation now share the same status. On the other hand, switching from one system of units to another becomes problematic - and the question of systems of units was one of the major concerns of Fourier's time, along with metric reform. This difficulty is not so acute when relationships between quantities are expressed in terms of double proportions. On the other hand, if we conceive of the quantities that enter an equation as simple numbers, what remains of this equation when some of them change? The answer to this question, that the others must also change, forms the basis of Fourier's discussion. This led him to redefine the very concept of homogeneity. Indeed, since he chooses to remain within the context of numerical equations, the principle of homogeneity as conceived in previous centuries is rendered meaningless, since what is being "compared" are no longer the physical quantities themselves. Fourier's concept of "dimension" thus undergoes a change of meaning: it no longer refers to what characterizes the *nature of* a quantity, such that two quantities are considered homogeneous ("of the same nature") when they have the same *dimension*, but simply to the power to which the conversion factor between two different units of the same fundamental quantity must be set, so that the numerical values of a given physical quantity can retain the same relationships with the other quantities in the two systems of units. But whether or not the latter definition is the same as the former is by no means obvious. The tension between these two conceptions of homogeneity and dimensions was to be present in most debates on the subject throughout the period under consideration, sometimes consciously, most of the time underlying it.

Chapter 2: The contributions of Gauss and Weber

2. 1 - Introduction

As we have seen, Fourier had two motivations for tackling the question of dimensions. The first - which can be described as traditional - is that the requirement for homogeneity constitutes a means of verification: ensuring the homogeneity of the two members of an equation expressing a physical law enables the detection of possible errors in the law itself. The second, which is one of Fourier's innovations, is that dimensional considerations make it possible to determine by which conversion factors the values of derived quantities must be multiplied for a given equation to remain correct, knowing the conversion factors for the fundamental quantities.

As far as the first approach is concerned, it's difficult to get a precise idea of how successful it was at the time. In addition to Poisson's work, which may have been inspired by Fourier's 1811 memoir,[1] it is conceivable that many of his contemporary readers appreciated his advice and sometimes used dimensional analysis for verification purposes. If this was the case, apart from Poisson's, such an approach seems to have remained part

[1] See chapter 1.

of the scholars' personal work, without being mentioned in publications as an argument in favor of the idea that this or that equation is indeed correct.

Fourier's second approach became increasingly popular from the 1860s onwards. It's not clear why dimensional analysis didn't arouse interest before this period. It would seem that the scientific community simply didn't see the point, until finally, some forty years after the publication of Fourier's treatise, the need to unify the different systems of units, which was of great concern to scholars at the time, particularly in Germany and Great Britain, revived the question of dimensions. However, prior to this revival, Carl Friedrich Gauss and Wilhelm Weber had made considerations which, while not involving the concept of dimension in the strict sense of the term, seemed to be similar to those evoked by Fourier concerning units.

Gauss and Weber carried out their work at a time when the problem of measurement precision was becoming increasingly important to scientists, particularly in Germany. In the 18th century, the notion of measurement precision was associated with an ideal of mathematical accuracy.[2] This situation began to change in the early 19th century.[3] The notion of precision was gradually dissociated from that of the mathematical ideal. One aspect of this reformulation was the use of probability calculus.

As early as 1809, Gauss asserted that minimizing uncertainty in results was one of the most important tasks of mathematics applied to the natural sciences. Although some data processing procedures already existed in the eighteenth century, it wasn't until the early nineteenth that a rigorous approach emerged. In Germany, it was Gauss himself who developed the method of least squares. However, it wasn't until the 1830s-1840s that its

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[2] Widmalm 1990: 179-206; Golinski1995 in Wise 1995: 72-91.
[3] Olesko1995 in Wise 1995: 103-134; Darrigol 2000: 42-76.

use became systematized and extended to other disciplines such as chemistry, mineralogy and physiology.[4] The ability to take random errors into account became one of the keys to quality results. [5]

A second aspect of measurement concerns standards. This concern extends beyond scientific studies: maintaining the metric system after the Napoleonic Wars prompted several German states to introduce weights and measures reforms. The reliability of measurements for commercial needs was a major concern for these states, which turned to scientists to establish standards. As early as 1816, the Prussian government commissioned members of the *Akademie der Wissenschaften* for this purpose. From 1836, Gauss himself was in charge of determining weights and lengths in Hanover. The definition of standards, whether for commercial or scientific purposes, is seen as requiring not only quality instruments, but also compliance with protocols, both in the replication of material standards and in the taking of the measurements themselves. Taking errors into account remains one of the most important quality criteria.

2. 2 - Gauss's absolute measurements

It was in the context of a project for absolute measurements that Gauss entered into considerations reminiscent of those of Fourier, though without involving changes in units. As early as the 1830s - a decade after Fourier's work - Gauss turned his attention to the problem of the units of derived quantities. For him, it was not a question of studying the conditions under which numerical equations remain invariant by changing units, but of

[4] Gauss 1809; Gauss1821, in Gauss 1880: vol.4, 1-108; Gauss 1816: 185-96. For the method developed by Legendre, see Legendre 1805: 72-80.
[5] Olesko1995 in Wise 1995: 103-134; Darrigol 2000: 42-76.

defining new units for derived quantities. He turned his attention to magnetic phenomena, and realized that the measurements made to date in this field were unreliable, as there was no constant standard. Henry Gambey's compass was used as a standard. However, there is no evidence that the needle's magnetic moment is constant: on the contrary, it should certainly be decreasing. At the same time, it is impossible to control these variations by choosing a reference location and placing the compass there, since the earth's magnetic field also varies.

In order to escape this circularity, Gauss suggests measuring magnetic phenomena by means of experiments which, ultimately, allow magnetic units to be expressed in terms of units whose stability is not in doubt. The way he achieves this goal is based on a certain conception of physical quantities. He considers it possible to reduce any quantity to those of mechanics. In particular, he proposes to reduce magnetic quantities to length, mass and acceleration (or length, mass and time; the two approaches being equivalent, since measurements of acceleration can be reduced to measurements of length and time). Before analyzing an example of this approach, it's worth clarifying a few expressions.

Gauss - and Weber after him - spoke of "fundamental measurements" and "absolute measurements." The former refers to measurements of the quantities to which he intends to reduce the others: length, mass and time (or acceleration).[6] The second refers to measurements of derived quantities whose units can be reduced to the units of his fundamental quantities. Gauss's concept of fundamental measurements is immediately

[6] This brings us back to the quantities used by Lazare Carnot in 1803: mass, length and time. Carnot's approach prefigures that of Gauss, in that it also consists in "reducing" other quantities (mechanical quantities in his case) to these three fundamental quantities. But the choice of these three quantities is quite natural, both because they correspond to the fundamental Newtonian concepts of time and space, and because they are the most commonly measured quantities.

reminiscent of Fourier's concept of fundamental units, in that it consists in reducing certain quantities to other quantities considered more fundamental, thus establishing a kind of hierarchy between physical quantities. The question, then, is to what extent and in what way these two concepts differ and in what way they converge.

There are two main differences between the two concepts. The first difference lies in the choice of quantities considered fundamental. In Gauss's case, these are necessarily mechanical quantities. This is not always the case with Fourier: as we have seen, he also uses temperature. This difference is due to the fact that Gauss believes that any quantity can be expressed by means of mechanical quantities, and that it is necessary to demonstrate this dependence. Such a position boils down to the view that all physical phenomena are ultimately mechanical in nature. By contrast, Fourier's choice of fundamental quantities is phenomenological: they are the quantities he is led to measure in practice as part of the experiments by which he explores the phenomena in question.

The second difference lies in the approach taken by the two scientists. For Fourier, because it is primarily a question of considering the problem of conversion in order to study invariance considerations, the fundamental quantities are those whose conversion factors need to be known when changing unit(s), in order to determine by which conversion factors all the numerical values of all the other quantities involved in a (numerical) equation must be multiplied for the equation to remain valid. For Gauss, it's a question of defining units whose validity is guaranteed by the fact that they can be reduced to other units that are beyond suspicion (in addition to the mechanistic considerations already mentioned above). The relationship between derived and fundamental quantities is established by absolute measurement. It is therefore also an experimental procedure.

The latter certainly corresponds to a formal expression, i.e. an equation that defines the derived quantity as a function of fundamental quantities (or quantities closer to the fundamental quantities in the reduction process). But whereas Fourier focused on such an expression, Gauss and Weber were primarily interested in the experimental procedure. What's more, since they were concerned with establishing units, every physical quantity involved had to have the value of unity. For example, the absolute unit of electromotive force is defined as follows :

> The absolute unit of measurement of the electromotive force is *that electromotive force which the unit of measurement of terrestrial magnetism exerts on a closed conductor, when the latter is oriented in such a way that the area of its projection on a plane perpendicular to the direction of terrestrial magnetism increases or decreases by one unit of area in one unit of* time.[7]

Despite these differences, Gauss and Weber sometimes offer discussions very close to Fourier's. This is particularly true when they deal with the problem of conversion between systems of units. This is particularly true when they discuss the problem of conversion between systems of units. Such a discussion is first found in Gauss's 1833 paper, "Mesure absolue de l'intensité du magnétisme terrestre" ("Absolute measurement of the intensity of terrestrial magnetism"), published in French in 1834.[8] In it, Gauss sets out to define an absolute unit for the intensity of the magnetic field, but he also addresses the question of changing units.

Once again, Gauss does not develop the concept of dimension. In this sense, his discussion is far less theoretical than that of Fourier, for whom the problem of conversion

[7] "As an absolute unit of measure of electromotive force, may be understood that electromotive force which the unit of measure of the earth's magnetism exerts upon a closed conductor, if the latter is so turned that the area of its projection on a plane normal to the direction of the earth's magnetism increases or decreases during the unit of time by the unit of surface." Weber 1861: 227.
[8] Gauss original Goettingue 1833, French translation1834, 5-69.

appeared in the context of a concern to ensure the invariance of equations, which led him to redefine the very concept of dimension and homogeneity. In comparison, Gauss's discussion seems quite limited. And yet, it's still a question of determining the numerical factor by which to multiply a derived quantity when changing the unit system. It's interesting to compare the approaches of the two scientists.

First of all, let's take a look at the notation used by Gauss. He uses two types of symbol: upper case and lower case. Capital letters refer to the units of the quantities under consideration.[9] When they belong to the unit system in which a conversion is to be made, Gauss indicates this with an apostrophe. Lower-case letters represent numerical conversion factors between two units of the same quantity. Since Gauss always considers the case of quantities whose numerical value is 1 in the initial system, these conversion factors also happen to be the numerical value of the quantity in the second system. For example, Gauss writes S = s S', where S refers to the initial unit of time, S' to the unit of time to be converted, and s is the conversion factor between the two units.[10] All the equations he uses in the passage we're interested in involve only lower-case symbols, since they concern the numerical values of quantities in a given system of units.[11] They are therefore purely numerical equations. This notation is of particular interest to us: consciously or not, it embodies the principle of homogeneity. In fact, by clearly distinguishing between numerical values and units - i.e. physical quantities considered as having a unit value - we make explicit the fact that all the quantities we're working with are purely numerical; we

[9] Gauss 1834: 66.
[10] Gauss 1834: 67.
[11] Gauss 1834: 67.

can't therefore violate the principle of homogeneity, since symbols representing physical quantities don't come into play.

The origin of Gaussian notation is unclear. John Roche offers an interesting speculation. He notes that some chemists, notably Antoine de Lavoisier in 1762, John Dalton (1766-1844) in 1805, Jacob Berzelius (1779-1848) in 1814, had already employed what he calls a "calcul de grandeurs." Thus Berzelius represents sulfuric acid by the expression "S + 3 O." Dalton writes "2 atoms of A + 1 atom of B = 1 atom of E." [12]

Roche notes:

> I don't know whether Gauss was influenced by contemporary chemistry in his choice of capital letters to represent physical units, for example. In chemistry, of course, capitals represent units of composition rather than standards of measurement.[13]

What's more, as Roche notes, in Gauss's notation, in an expression such as S = s S' mentioned above, the equal sign indicates that the term on the left and the term on the right represent the same physical state. In Gauss, as in chemistry, however, we are dealing with concrete quantities. Measurement standards are concrete units. In Dalton's equation, the symbols A, B and E refer abstractly to substances. But the terms of his equation do represent concrete quantities; and in Berzelius, the symbols S and O represent concrete quantities.

[12] "2 atoms of A + a tom of B = 1 atom of E," Roche 1998: 199.
[13] "I am unaware whether Gauss was influenced y contemporary chemistry in his choice, for example, of capital letters for his physical units. In chemistry, of course, the capitals represented units of composition rather than standards of measurement." Roche 1998: 199.

Gauss begins by showing how to reduce the unit of magnetic intensity to the fundamental units. The equations that enable him to do this are the ones he then uses to show how a numerical value k representing magnetic intensity in a first system of units is affected when the fundamental units are changed, taking on the value k'.

In Gauss's work, the verbal explanation of the steps involved in reducing the unit of intensity to the fundamental units is found before the list of equations in question, but for the sake of clarity we'll list them simultaneously here.

Gauss states:

> To establish the unit [of magnetic field] V, we start with the unit of free magnetism M and the unit of distance R, and take the force V equal to the force of magnetism M at distance R.[14]

It uses :

$$v = \frac{m}{r^2} \tag{5}$$

with v corresponding to the magnetic field intensity, m the quantity of magnetism and r the distance of the point considered from this mass.[15]

Gauss continues:

> For the unit M we adopt the quantity of magnetic fluid which, acting on an equal quantity M placed at a distance R, produces a driving force (or if we prefer, a pressure) equal to that W which is taken as the unit, i.e. equal to the force exerted by the accelerating force A [Gauss thus designates acceleration] taken as the unit acting on the mass P, also taken as the unit.[16]

Or :

$$\frac{m^2}{r^2} = w = pa \tag{6}$$

with r now corresponding to the distance between two magnetic quantities, w to the magnetic force between them, a to acceleration and p to (inertial) mass.

[14] Gauss 1834: 65-66.

[15] As mentioned above, these symbols correspond to the numerical values of the quantities in the system to be converted.

[16] Gauss 1834: 66.

As for the force [acceleration] A taken as a unit, there are two ways of defining it: it can be deduced either from an immediately given analogous force, such as gravity in the place of observation, or from its effect on bodies to set them in motion. In the second procedure, which we have followed in our calculations, two new units are required, namely: the unit of time S and the unit of speed C, so that the force A taken as the unit is the accelerating force which, acting during time S, produces speed C. [17]

Thus :

$$a = \frac{c}{s} \tag{7}$$

where c is speed and s is duration.

Finally, this velocity C itself is the velocity that responds to a space R traversed by uniform motion during time S.
So it's clear that unit V depends on three units, either R, P and A, or R, P and S.[18]

In other words:

$$c = \frac{r}{s} \tag{8}$$

where r corresponds to the space covered and s to the duration.

Gauss wanted to obtain an expression for k' (the numerical value of the intensity in the second system) as a function of k (its numerical value in the first system), and as a function of the conversion factors of the fundamental units - exactly the problem that Fourier was facing. His starting point was the definition of k and k' :

$$kV = k'V' \tag{9}$$

and represents the conversion factor between intensity units in the two systems by v :

$$V = vV' \text{ or } , \frac{V}{V'} = v \tag{10}$$

therefore :

$$kv = k'. \tag{11}$$

[17] Gauss 1834: 66.
[18] Gauss 1834: 66.

The key is to determine v in terms of the conversion factors of the fundamental units, which Gauss does by combining the equations that reduce V *to* the fundamental quantities. The result is :

$$\frac{m^2}{r^2} = p\,a \qquad \Leftrightarrow \qquad m = r\sqrt{pa}\,, \tag{12}$$

therefore :

$$v = \frac{m}{r^2} = \frac{r\sqrt{pa}}{r^2} = \frac{\sqrt{pa}}{r} = \sqrt{\frac{pa}{r^2}} \quad . \tag{13}$$

Since $a = \dfrac{c}{s}$:

$$v = \sqrt{\frac{pc}{r^2 s}} \quad . \tag{4)14}$$

And since : $c = \dfrac{r}{s}$

$$v = \sqrt{\frac{pr}{r^2 s^2}} = \sqrt{\frac{p}{r\,s^2}}\,. \tag{15}$$

Gauss thus obtains :

$$k' = k\sqrt{\frac{p}{rs^2}}\,. \tag{16}$$

We can see that Gauss is faced with a problem quite similar to that of Fourier. Remember that Fourier dealt with each fundamental unit separately, determining the dimension exponent corresponding to each of them. Gauss, on the other hand, treats the problem as a whole, focusing on the various quantities involved in the equation defining the quantity of interest, and expressing these quantities in terms of quantities closer to the fundamental ones, and so on until he arrives at the latter. In this way, he derives an expression that contains all the fundamental units at once. But he doesn't bother to define the dimensions of the sought-after quantity corresponding to each of these fundamental units.

From the purely practical point of view of converting quantities from one system of units to another, the two methods amount to the same thing. Where Fourier would say that the magnetic field intensity has dimensions in relation to mass, $1/2$, in relation to length, $-1/2$, and in relation to time, -1 , Gauss says $k' = k\sqrt{\frac{p}{rs^2}}$ (except that Fourier doesn't mention the possibility of fractional-dimension exponents). On the other hand, the two approaches differ conceptually: whereas Gauss is strictly concerned with the problem of changing units, Fourier is motivated by a theoretical concern for the (numerical) equations themselves, so that for him it's a matter of studying the conditions under which they remain invariant under changing units. For this reason, Fourier was led to define the concept of dimension exponent, while Gauss felt no need to do so.

As John Roche points out, this led Gauss to use exponents of fractional dimensions. This was not the case with Fourier. There was no need to do so in the context of heat theory.[19]

2. 3 - Wilhelm Weber's contributions

Almost fifteen years after Gauss's discussion, in 1851, we find Weber's reasoning even closer to Fourier's considerations.[20] Admittedly, Weber still doesn't use Fourier's concept of dimension, but he does come close.[21]

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[19] Roche 1998: 193.
[20] The article in question was first published in *Annalen der Physik*, vol. lxxxii, 337; it was then translated into English ten years later by E. Atkinson, and published under the title "On the measurement of Electric Resistance according to an absolute standard," *Philosophical magazine*, series 4. The references given here are for this translation.
[21] Of Weber's many publications, this one interests us because it was translated into English and published in 1861 in the pages of *Philosophical magazine*.

Weber develops the ideas that interest us here in the context of determining an absolute unit for electrical resistance. He uses a notation similar to that of Gauss: a capital letter represents a unit quantity of the quantity in question, a small letter represents the number of times this unit quantity is included in the quantity. For example, he says that *tT* is the earth's magnetic field, *sS* is a duration, etc.[22] He does not use lower-case letters as conversion factors, as he is not led here to use such factors as Gauss did.

What brings the present work closer to Fourier's is that Weber sets out to determine which changes in more fundamental units do or do not affect the unit of resistance. To understand Weber's reasoning, we first need to see how he advocates defining the unit of resistance. Weber is opposed to any definition involving the resistivity of the material used, because of its variability from one wire to another, whatever the purity of the metal. He therefore proposed defining an absolute unit of resistance based on absolute units of electromotive force and current intensity. Contrary to Gauss (and Fourier), Weber does not propose to reduce resistance measurement entirely to fundamental measurements - indeed, he says this is unnecessary:

> To measure these quantities [electromotive force and current], no particular fundamental measurement is required.[23]

However, since the measurements of the derived quantities he employs are absolute, it is obvious that it remains possible to do so - and this possibility is what qualifies the resistance measurement as absolute.[24] This is how he defines the absolute unit of electromotive force:

> The absolute unit of measurement of the electromotive force is *that electromotive force which the unit of measurement of terrestrial magnetism exerts on a closed*

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[22] Weber 1861: 228.
[23] "In measuring these quantities [electromotive force and intensity], no specific fundamental measures are requisite." Weber 1861: 227.
[24] Weber states: "The measures of electric resistances can be referred to an absolute standard, provided measures of *space*, *time* and *mass* are given as fundamental measures"; Weber 1861: 227.

conductor, when the latter is oriented in such a way that the area of its projection on a plane perpendicular to the direction of terrestrial magnetism increases or decreases by one unit of area in one unit of time. [25]

This definition can be represented as follows:

Definition of the absolute unit of electromotive force :
Electromotive force exerted by one unit of the Earth's magnetic field on a conductor oriented as shown below.

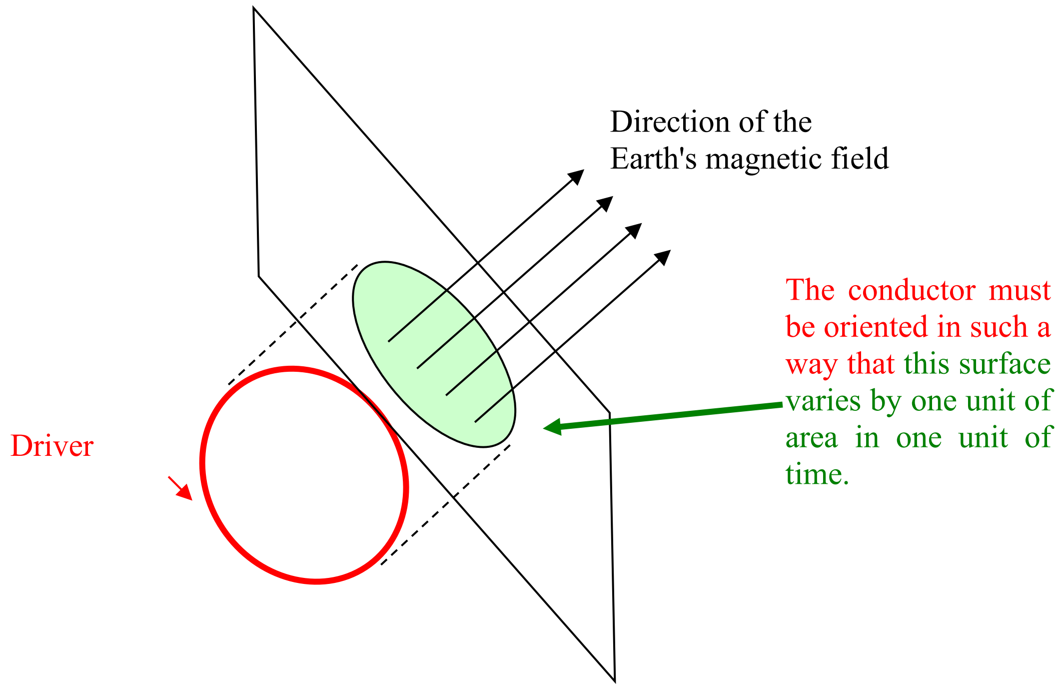

Figure 2.1

This corresponds to the relationship :

$$e = \frac{f\,t}{s}, \qquad \textbf{(17)}$$

with :

e: electromotive force exerted by the magnetic field on the conductor
f: area enclosed by the conductor
t: intensity of the earth's magnetic field

[25] "As an absolute unit of measure of electromotive force, may be understood *that electromotive force which the unit of measure of the earth's magnetism exerts upon a closed conductor, if the latter is so turned that the area of its projection on a plane normal to the direction of the earth's magnetism increases or decreases during the unit of time by the unit of surface*." Weber 1861: 227. Italics in original.

> s: time required for the conductor to rotate from an orientation parallel to the earth's magnetic field, to an orientation perpendicular to this field.

Weber defines the absolute unit of current intensity as follows:

> By absolute unit of measurement of intensity, we mean *the intensity of a current which, when flowing through a plane whose magnitude is the unit of measurement, exerts, in accordance with electromagnetic laws, the same action at a distance as a long magnet containing one unit of measurement of magnetism.*[26]

which corresponds to equation :

$$i = \frac{m}{f}, \tag{18}$$

with :

i: current intensity in the conductor
f: area enclosed by the conductor
m: amount of magnetism of a magnet which, in place of the circuit, would exert the same force as the conductor

Weber does not deal directly with absolute measurements of magnetism and magnetic field, but refers the reader to Gauss's treatise on the subject.[27] It may seem strange that Weber should attempt to define the unit of derived quantities on the basis of a measurement, albeit an absolute one, of the earth's magnetic field, all the more so since the fact that the latter varies over time in the same place was, we recall, one of Gauss's motivations for developing more satisfactory magnetic units than those then in existence. But this approach didn't really pose a problem, as Weber went on to demonstrate that the numerical value of resistance was unaffected by a change of magnetic field unit.

In addition to the two equations above, Weber gives us Ohm's law:

$$w = \frac{e}{i}, \tag{19}$$

[26] "As an absolute unit of intensity, can be understood *the intensity of that current which, when it circulates through a plane of the magnitude of the unit of measure, exercises, according to electro-magnetic laws, the same action at a distance as a bar-magnet which contains the unit of measure of bar magnetism*." Weber 1861: 227. Italics in original.
[27] Gauss 1833: 241 and 591.

with :

w : resistance
i: current intensity in the conductor
e: electromotive force in the conductor

By substituting the expressions for e and i given by the first two equations into the latter, Weber obtains an expression for resistance as a function of the earth's magnetic field and a magnet's field:

$$i = \frac{m}{f} \ , \ e = \frac{f\,t}{s} \tag{20}$$

$$w = \frac{e}{i} = \frac{f\,t}{s} \cdot \frac{f}{m} = \frac{ff\,t}{s\,m}. \tag{21}$$

f: area enclosed by a conductor
t: intensity of the earth's magnetism
m: intensity of a magnet's magnetism
s: time taken for the conductor plane to rotate from an orientation parallel to the direction of the earth's magnetic field to an orientation perpendicular to it.[28]

Weber continues:

> The constancy of the unit of measurement of electrical resistance is thus guaranteed as long as the four measurements in question (space, time, and the units of measurement of terrestrial magnetism and the magnetism of a magnet) undergo no change. But this in no way implies that the conservation of these four measurements is a necessary condition for the constancy of the unit of measurement of electrical resistances; simply maintaining the unit of measurement of velocities is sufficient for this.[29]

To demonstrate this, Weber gives himself the following equation, which he borrows from Gauss's treatise:[30]

[28] Weber 1861: 228.
[29] "The unchangeability of the unit of measure for electric resistance can accordingly be guaranteed so long as the four given measures (space, time, and the units of measure for the earth's magnetism and for bar magnetism) are obtained unchanged. But it by no means follows that the maintenance of these four given measures is a necessary condition for the unchangeability of the unit of measure of electric resistances; the simple maintenance of that unit of measure for *velocities* is sufficient for the purpose." Weber 1861: 229.
[30] Gauss 1833.

$$t = \frac{m'}{r^3}, \qquad (22)$$

with :

t: intensity of the earth's magnetic field

m': amount of magnetism of a magnet whose magnetic field would be the same as that ofla Terre (whose axis must therefore be oriented parallel to the Earth's magnetic field).

r: distance between the center of the magnet and the center of the plane formed by the conductor, where the line between the two is perpendicular to the direction of the Earth's magnetic field.

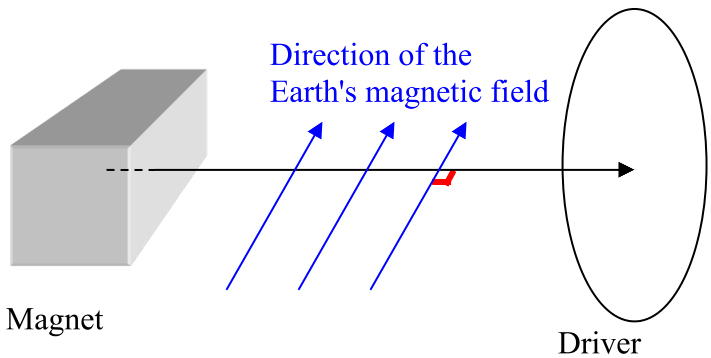

Figure 2.2

Substituting this expression for t into the previous equation gives :

$$w = \frac{f\,f}{s\,m} \cdot \frac{m'}{r^3}, \qquad (23)$$

plus :

$$f = r'r', \qquad (24)$$

where r' is the length of the side of a square with the same area as the surface formed by the conductor.

Substituting this expression for f into the above equation for w, we get :

$$w = \frac{r'^4}{s\,m} \cdot \frac{m'}{r^3}, \qquad (25)$$

which Weber writes as :

$$w = \frac{r'^3}{r^3} \cdot \frac{m'}{m} \cdot \frac{r'}{s} \tag{26}$$

If we then change units, we can see that the numerical value of the expression $\frac{r'^3}{r^3} \cdot \frac{m'}{m}$ will not be affected (since it is made up of quotients of homogeneous quantities). Only the numerical value $\frac{r'}{s}$ will change. The numerical value of resistance will therefore be affected by a change of units in the same way as this expression, which Weber identifies with a velocity. Weber concludes:

> If the measurement [i.e. unit] of speed is taken *n* times larger or smaller, the unit of measurement of resistance simultaneously becomes *n* times larger or smaller. [31]

We can see, then, that Weber's approach to the problem of the impact of a change of units on the absolute unit he sets out to define is very close to that of Gauss in the work previously studied, but is also slightly different. Like Gauss, Weber sought a numerical equation expressing the value of the quantity of interest in terms of quantities closer to the fundamental quantities. However, unlike Gauss, he does not seek to have this equation directly express the relationship between this quantity and the fundamental quantities, as Gauss does $v = \sqrt{\frac{p}{r s^2}}$. Weber, on the other hand, tries to highlight several physical quantities corresponding to fundamental quantities (for example, several lengths r and r'), so as to form quotients of homogeneous quantities whose unit we know will not influence the unit of the derived quantity of interest. In this respect, Gauss's approach is more structured: it involves a process of successive reduction, replacing at each stage the

[31] "if the measure of velocity is taken *n* times larger or smaller, the unit of measure for resistance becomes simultaneously *n* times larger or smaller." Weber 1861: 230.

remaining derived quantities in an equation by quantities closer to the fundamental ones, until only the latter remain. In this way, the result obtained is clearly unique. Weber, on the other hand, is left with the impression that, after all, he could just as easily have used another equation than $t = \frac{m'}{r^3}$ in his demonstration, and would then have arrived at a result that is certainly compatible, implying that the unit of resistance varies like a velocity during a change of system, but nevertheless different in detail - i.e. with an equation other than $w = \frac{r'^3}{r^3} . \frac{m'}{m} . \frac{r'}{s}$. In this respect, Gauss's reasoning seems closer to Fourier's than Weber's: for each derived quantity considered, there is only one way of obtaining a result (in this case, the quantity's dimensional exponents), and this is unique in that the result is given in terms of each fundamental quantity.

Related to this question is the fact that Weber's result retains the exact physical meaning of the quantities contained in the equation: r remains the distance illustrated in Fig. 2, r' always represents the length of the side of a square of the same area as the surface formed by the conductor, and so on. These are all quantities that characterize a specific type of physical system. Gauss, on the other hand, obtains an expression which, containing only one representative of each fundamental quantity, brings them into play in a more abstract way, as it were, in $v = \sqrt{\frac{p}{r s^2}}$, r is not the distance travelled by a precise system under well-defined conditions, nor s the corresponding duration, etc. The equation $v = \sqrt{\frac{p}{r s^2}}$ simply gives the numerical value that the magnetic field intensity must take on when the fundamental units are changed, so that a mass that previously had the numerical value1 a now has the numerical value *p* in the new system, and so on. In this respect too, Gauss's approach is closer to Fourier's than Weber's, since Fourier's fundamental quantities do not correspond to precise physical systems either, but share the more abstract aspect of *p*, *r* and *s* with Gauss.

Nevertheless, Weber's work is close to Fourier's in one essential respect: Weber directly relates the conversion coefficient of his (more) fundamental quantity to that of its derived quantity - a coefficient that is the same in both cases: if the unit of velocity becomes *n* times greater, the unit of resistance is also multiplied by *n*. Only two points separate him from Fourier: he does not express this dependence as a function of each fundamental quantity separately, and he does not develop the concept of dimension exponent. The latter is due to the fact that, having taken velocity as the fundamental quantity, *n* simply has 1 as its exponent.

As Roche notes:

> The simplicity of Weber's dimensional result contrasts with the complexity of his measurement equation. This clearly highlights the significant loss of information about the measurement process that occurs when the algorithm linking an indirectly measured quantity to the basic quantities is reduced to the algorithm linking the derived conversion factors to the basic conversion factors.[32]

The theme of loss of information, between the definition of a derived quantity and its dimensional formula, will recur in debates on the meaning of dimensions.[33]

In addition to the article we've just discussed, Weber was also responsible for contributions that went beyond dimensional issues, but to which these were closely linked

[32] "The simplicity of this dimensional result of Weber's contrasts with the complexity of his measuring equation. It brings out vividly how much information about the process of measurement is lost when the algorithm relating an indirectly measured quantity to the base quantities is reduced to the algorithm relating derived conversion factors to base conversion factors." Roche 1998: 201.
[33] See chapter 10, 327.

in France and Great Britain. In the 1840s and 1850s, he defined several systems of units for electrical and magnetic quantities, some of which were gradually adopted in the countries we are interested in. He proposed the so-called "electrodynamic," "electromagnetic" and "electrostatic" systems.[34] The first is based on Ampère's formula:

$$d^2 f = ii' \frac{ds\, ds'}{r^2} \left(\sin\alpha \ \sin\beta \cos\gamma - \frac{1}{2} \cos\alpha \ \cos\beta \right)$$

where f is the force between two current elements, i and i' are the current intensities in these elements, ds and ds' their lengths, r the distance between them, and ,$\alpha\beta$ and γ the angles characterizing their orientation. In the electrodynamic system, the unit of current is defined by this formula. The units of all other electrical quantities are then defined from this unit; in particular, the unit of charge is the quantity of charge displaced by one unit of current in one unit of time. [35]

The electromagnetic system is based on Coulomb's formula for the force between two magnetic poles $f = \frac{mm'}{r^2}$ and the equation giving the force between a current and a magnetic pole $f = \frac{ilm}{r^2}$ (where l represents the length of the wire).[36] Finally, the electrostatic system is based on Coulomb's formula for the force between two electric charges . $f = \frac{qq'}{r^2}$ [37] In both cases, the unit of charge and the unit of current are linked by $q = it$, and therefore by the unit of time.

[34] See Darrigol 2000: 399.
[35] Weber 1846: 51-60.
[36] Weber 1846.
[37] Weber 1850: 267-270.

As we'll see in the next chapter, the electromagnetic system and, to a lesser extent, the electrostatic system, spread to Great Britain and then France. The notion of absolute measurement also spread, as did the mechanistic notion that measurements of electrical and magnetic quantities could be expressed in terms of length, mass and time.

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Chapter 3: Maxwell's synthesis

3. 1 -Introduction

In the 1860s, dimensional considerations were finally taken up again in terms very similar to those of Fourier. The first references in this field come from Great Britain, and are due to James Clerk Maxwell (1831-1879) and telegraph engineer Fleeming Jenkin (1833-1885) in an 1863 publication.[1] They are linked to the desire to establish standards for certain electrical quantities, in particular resistance. This concern largely depended on British interest in the industrial applications of electricity, which were predominant in England and absent from Gauss and Weber's approach.[2] As we shall see, Maxwell's and Jenkin's conceptions are nonetheless close to those of Gauss and Weber on essential points.

3. 2 - The early 1860s: the British Association's Committee on electrical standards

In the early 1860s, the units used in electricity were very diverse. This situation made communication within the scientific community difficult: experiments were reported in *ad*

[1] Maxwell & Fleeming 1863: 130-176.
[2] Smith 1998, especially chapter 13. Olesko 1996: 118.

hoc units, which differed from country to country. For example, electromotive force is indicated in Daniell or Bunsen elements, in reference to the battery used. The situation is particularly intolerable when it comes to resistance. There are no less than a dozen different units: named after scientists or countries (Bréguet, Jacobi, German units...), or given in length of wire of a given diameter, or in height of mercury.[3] This situation posed problems for engineers and telegraphers alike. Telegraphy, the first industrial application of electricity, appeared in the late 1830s, first in England and the United States, then in France from the mid1840. A In the late 1850s, the first unsuccessful attempts to establish transatlantic lines were made - they finally bore fruit in 1866. This boom was accompanied by the emergence of a new professional category, telegraph engineers, for whom electrical measurements were a daily task, and for whom the absence of generalized conventions concerning units imposed onerous conversions, just as it did for scientists wishing to compare their results with those of their colleagues.[4] Such conversions are both a waste of time and a source of error.

To remedy this situation, in 1861 the *British Association for the Advancement of* Science, at the instigation of William Thomson, set up a committee to study the question of electrical units.[5] The initial objective was limited: to define a unit of electrical resistance. However, the committee soon extended its concerns to electrical measurements in general.[6] Most of the important decisions were taken in 1863, and are the subject of its second report.

[3] Atten 1992: 112-113.
[4] Atten 1992: 93-94.
[5] Darrigol 2000: 124-125.
[6] *Report of the 33rd meeting of the British Association for the Advancement of Science, held atNewcastle-upon-Tyne in August and September 1863,* London , John Murray, 1864, 111. The first reports were republished together in 1873, but with some modifications (*Reports of the Committee on electrical standards appointed by the British Association for the Advancement of Science*, London, 1873).

Dated August 26, the committee's members included William Thomson, James Prescott Joule (1818-1889), Fleeming Jenkin (to be discussed shortly) and James Clerk Maxwell.

In its 1863 report, the committee advocated the adoption of an absolute system of measurement, in which electrical and magnetic units would be derived from fundamental units of length, mass and time, as envisaged by Gauss and Weber. The Committee thus shared the mechanistic conceptions of the latter, and as we shall see, also took up Weber's idea of electrostatic and electromagnetic systems. As for the choice of fundamental units, Weber had favoured the metre, the milligram and the second, but the Committee eventually agreed with Thomson and opted for the centimetre, the gram and the second, hence the name "C.G.S. system". - although it was also referred to at the time as the "B.A. system" (for British Association). (for *British Association*).

As far as electrical units are concerned, the committee states that four electrical quantities are "capable of being measured": current, electromotive force, resistance and quantity of electricity. It undertook to define them on the basis of the following three equations:[7]

Ohm's law : $$C = \frac{E}{R} \ , \qquad \textbf{(27)}$$

where C is the current, E the electromotive force and R the resistance;

$$Q = C\,t \ , \qquad \textbf{(28)}$$

where Q is the quantity of electricity and t is time;

$$W = C^2\,R\,t \,, \qquad \textbf{(29)}$$

where W is the work "equivalent to all the effects produced in the circuit."

[7] B.A. Report 1864: 113-114.

As these relationships are not sufficient to define all the units in question, it is necessary to introduce others. The committee then presents two possible choices.

The first is to define the amount of static electricity using Coulomb's equation:

$$F = \frac{Q^2}{d^2}. \tag{30}$$

The unit of quantity of electricity is then that present on a sphere when a similar sphere placed at a unit distance is repelled with the unit of force. Hence the unit of current is determined by equation (2), the unit of resistance by equation (3), and the unit of electromotive force by (1). The result corresponds to Weber's system of electrostatic units, although he did not use equation (3).[8]

But this is not the system recommended by the committee:

> The electrostatic system we have just defined is not the one that will prove most useful in general. It is based on a static phenomenon, whereas the main application of electricity today is dynamic, and depends on electricity in motion, or voltaic currents with their accompanying electromagnetic effects. [9]

The Committee prefers the electromagnetic system, in which equation (4) plays no role, and for which the following equation is substituted:

$$f = \frac{CLm}{r^2}, \tag{31}$$

where f is the force between a magnetic pole and a circular wire, centered on the pole, in which a current of intensity C flows, L *is* the length of this wire, r is the radius of the circle[10] and m is the intensity of the magnetic pole. The unit of the latter has not yet been defined. The committee defines it by Coulomb's law, which gives the force between two poles:

[8] B.A. Report 1864: 114.
[9] "The electrostatical system just defined is not that which will be found most generally useful. It is based on a static phenomenon, whereas at present the chief application of electricity are dynamic, depending on electricity in motion, or on voltaic currents with their accompanying electromagnetic effects." B.A. Report 1864: 115.
[10] The report uses the symbol k for this quantity.

$$F = \frac{m^2}{d^2} \ . \qquad \textbf{(32)}$$

The unit of quantity of magnetism is that of a pole if it repels a similar pole, separated by one unit of distance, with a force of one unit. The Committee insists on the incompatibility of the two systems: "Equations (4) and (5) are incompatible with each other, if equation (2) is taken as fundamental."[11]

In addition, as the units of other quantities derived from these definitions proved inconvenient, being either too large or too small, the Committee defined fractions or decimal multiples, thus creating what became known as the "practical *units*" system.[12]

The Committee does not fail to include dimensional considerations. These are grouped together in Appendix C of the report. This appendix, by Maxwell and Fleeming Jenkin, is a veritable treatise. In terms of volume, it represents more than half of the report itself. In it, the authors justify the Committee's approach and explain their motivations. Fleeming Jenkin was one of the Kingdom's leading telegraph engineers. In 1859, while working on the first transatlantic telegraph cable - which only operated for a few weeks - he had befriended William Thomson, who, it should be remembered, was largely responsible for the Committee's existence. Jenkin had also taken part in submarine cable trials in the Mediterranean in 1855.

So it's hardly surprising that Maxwell and Jenkin's article begins with these words:

> The progress and growth of the electric telegraph has made a practical knowledge of electrical and magnetic phenomena necessary for many people who work in one way or another on the construction and maintenance of lines, and interesting for

[11] "Equations (4) and (5) are incompatible one with another, if equation (2) be considered fundamental"; B.A. Report 1864: 115.
[12] B.A. Report 1873: 31-32.

> many others who do not wish to remain ignorant about the use of the cable network that surrounds them.[13]

More broadly, they express the feeling that the definition of units is indispensable to the very possibility of science as a collective enterprise:

> All accurate knowledge is based on the comparison of one quantity with another. In much experimental research carried out by individuals, the absolute values of these quantities are unimportant; but whenever many people have to act together, they need to have a common conception of the measurements to be used.[14]

Maxwell and Jenkin's motivations are therefore in perfect harmony with the objectives of the committee of which they are members. They also defend the Committee's particular choice. They insist on the need to limit the arbitrariness inherent in any choice, by taking a system that is as much in line with nature as possible. This concern is motivated by the desire to have a system that is as rational and objective as possible, and at the same time better able to be accepted by all, whatever the country considered, because of its universal character:

> It [the system in question] is easy to learn; it simplifies all sorts of calculations; it is more readily accepted by the rest of the world; and it bears the seal, not of some legislator or man of science, but of nature.[15]

The criterion that the unit system should be "natural" operates on two levels.

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[13] "The progress and extension of the electric telegraph has made a practical knowledge of electric and magnetic phenomena necessary to a large number of persons who are more or less occupied in the construction and working of the lines, and interesting to many others who are unwilling to be ignorant of the use of the network of wires which surrounds them." Maxwell & Fleeming 1863: 130.

[14] "All exact knowledge is founded on the comparison of one quantity with another. In many experimental researches conducted by single individuals, the absolute values of those quantities are of no importance; but whenever many persons are to act together, it is necessary that they should have a common understanding of the measures to be employed." Maxwell & Fleeming 1863: 131.

[15] "It [the system suggested] is easily learnt; it renders calculation of all kinds simpler; it is more readily accepted by the world at large; and it bears the stamp of the authority, not of this or that legislator or man of science, but of nature." Maxwell & Fleeming 1863: 131.

Firstly, it led to the adoption of Gauss's mechanistic conceptions. Maxwell and Jenkin thought it possible to derive all other units, especially those of electrical quantities, from fundamental mechanical units, i.e. mass, length and time: [16]

> The phenomena through which electricity is known to us are mechanical in nature, and must therefore be measured by mechanical units or standards. [17]

The way in which derived units are obtained from *fundamental standards* or *primary units* can be described as natural, since it is determined by the relationships between these quantities themselves, relationships that reflect natural laws. Moreover, as with Gauss and Weber, these relationships are based on an experimental procedure translating an equation that defines the quantity in question:

> Our task is to explain how these units [those of electrical quantities] can be derived from elementary quantities; in other words, we will endeavor to show how all electrical phenomena can be measured solely in relation to time, mass and space, and mention in each case a concrete method for making the observation.[18]

These explanations take up a large part of the article.

Secondly, the fundamental units chosen had to be natural. The unit of length then in force in France is the metre, defined as one ten-millionth of the distance between the pole and the equator as calculated by Jean Baptiste Joseph Delambre (1749-1822), and the unit of mass is based on this unit of length, being the mass of one cubic centimetre of water

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[16] Maxwell used the same term again in his *Treatise of Electricity and Magnetism* , vol. 1, 1873, 2-4.
[17] "The phenomena by which electricity is known to us are of a mechanical kind, and therefore they must be measured by mechanical units or standards." Maxwell & Fleeming 1863: 131.
[18] "Our task is to explain how these units [i.e. those of electrical quantities] may be derived from the elementary ones; in other words, we shall endeavor to show how all electric phenomena may be measured in terms of time, mass and space only, referring briefly in each case to a practical method of effecting the observation." Maxwell & Fleeming 1863: 131.

at standard temperature.[19] The unit of time - the second of mean solar time - is obtained by dividing (in the sexagesimal system) the duration of the mean day. [20]

Maxwell and Jenkin devote the final pages of their article to dimensional analysis. As with Fourier, knowing to what power the measurements of the fundamental quantities intervene in the measurements of the derived quantities appears to be a means of performing conversions between systems:

> Every measurement we deal with involves measures of time, space and mass alone as factors; but these measures are sometimes involved to one power, and sometimes to another. To move from one set of fundamental units to another, as well as for other purposes, it's useful to know to what power each of these fundamental measures intervenes in the derived measure. [21]

For example, when it comes to strength, they state:

> The dimensions of a force are $\left[\frac{LM}{T^2}\right]$; in other words, if we wish to switch from the English to the French system of measurement, the French unit of force will be the same as the English unit $\frac{3.28\times15.43}{1}:1$, or as 50.6 is 1; because there are 3.28 feet in a metre, and 15.43 grains in a gram. [22]

Maxwell and Jenkin's approach is thus a direct heir to Fourier's. Although they didn't mention the latter in 1863, in his famous treatise *Treatise on Electricity and Magnetism*,

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[19] The temperature in question was that corresponding to the maximum density of water (i.e. around $4^{(o)\,C}$), but the authors do not specify this, nor do they allude to a standard temperature. Furthermore, the authors do not explicitly advocate the use of these units. They mention them after referring to British units, and indicate their value in relation to the latter. Maxwell & Fleeming 1863: 131-132.

[20] Maxwell & Fleeming 1863: 131-132.

[21] "Every measurement of which we have to speak involves as factors measurements of time, space, and mass only; but these measurements enter sometimes at one power, and sometimes at another . In passing from one set of fundamental units to another, and for other purposes, it is useful to know at what power each of these fundamental measurements enters into the derived measure." Maxwell & Fleeming 1863 : 132.

[22] "The *dimensions* of a force are $\left[\frac{LM}{T^2}\right]$; in other words, if we wish to pass from the English to the French system of measurements, the French unit of force will be to the English as $\frac{3.28\times15.43}{1}:1$, or as 50.6 to 1; because there are3.28 feet in a metre, and 15.43 grains in a gram." Maxwell & Fleeming 1863: 132.

Maxwell would explicitly refer to Fourier's discussion in the *Theory of Heat* as the origin of dimensional theory.[23] It's likely, then, that Maxwell and Jenkin's discussion was already inspired by Fourier's.

We may wonder whether Maxwell and Jenkin share Fourier's views on the importance of dimensional analysis from a theoretical point of view. It should be remembered that Fourier went so far as to compare his dimensional principles to fundamental lemmas in geometry, and that his interest in the very question of conversion was of a theoretical nature, since it was a question of guaranteeing the invariance of numerical equations (by changing units). Although this concern for invariance is not as explicit in Maxwell's and Jenkin's work, it seems clear that their motivations are partly theoretical, since they state that all science is based on the comparison of quantities, and therefore on measurement.[24] Moreover, they conceive of dimension formulas as the direct expression of relationships between physical quantities themselves:

> The value of a force is proportional to length and mass, but inversely proportional to the square of a time. *This is expressed by saying that* the dimensions of a force are . $\frac{LM}{T^2}$ [25]

This quotation suggests that for them, dimension formulas reflect the very nature of the physical quantity. This is all the more likely given that they used the expressions "the dimensions of the unit of..."[26] , and "the dimensions of [the physical quantity itself]."[27]

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[23] Maxwell 1873: vol.1, footnote.
[24] Maxwell & Fleeming 1863: 131.
[25] Our italics. "[...] the value of a force is directly proportional to a length and a mass, but inversely proportional to the square of a time. This is expressed by saying that the dimensions of a force are $\frac{LM}{T^2}$;...," Maxwell & Fleeming 1863: 132.
[26] For example, "The dimensions of the unit magnetic pole," "unit of magnetization," or "the unit of magnetic potential" Maxwell & Fleeming 1863: 133-135.
[27] For example, "the dimensions of H," "the dimensions of q" Maxwell & Fleeming 1863: 133 and 149 respectively.

In terms of terminology and notation, the way Maxwell and Jenkin express themselves is slightly different from that of Fourier. The latter indicated separately, for each fundamental unit, the exponent to which it must be raised to convert a given derived quantity. He thus defined several "dimension exponents", one for each fundamental unit. Maxwell and Jenkin, on the other hand, give an integer expression for each derived quantity, an expression that includes the three fundamental quantities (represented by capital letters) to the power of what Fourier called their dimension exponent. For example, for force we have the expression $\frac{LM}{T^2}$. Here, L represents length, M mass and T duration. Maxwell and Jenkin use the term "dimensions" (plural), to designate the whole of this expression. Unlike Fourier, they do not use an expression such as "dimension exponent." They also leave it implicit that what needs to be raised to the appropriate power when converting between systems of units are the conversion factors themselves - but they make the point clear with examples.

As John Roche points out, Maxwell's and Jenkin's dimension formulas appear as algorithms linking derived units to fundamental units. In contrast, Fourier's formulas are concerned with the relationship between *conversion factors* for derived units and *conversion factors* for basic units, not with the units themselves. With Gauss, it was the relationship between the measurements of derived quantities and their concrete units, or the relationship between the units of the same physical quantity. [28]

When the report was republished in 1873, a number of notational additions were made. In particular, quantities are represented by the product of the numerical value of their measurement and the concrete unit. The first is represented by a single letter, the second

[28] Roche 1998: 206.

by the same letter enclosed in square brackets: Q [Q], for example. This notation is very similar to Weber's: square brackets are used to represent unit quantities, whereas Weber used capital letters. The units of the fundamental quantities are thus represented by [L], [M] and [T].[29] As a consequence, square brackets are used to indicate the dimensions of a derived quantity. Thus we find expressions such as "the dimensions of [C],"[30] "the dimensions of a force are $\left[\frac{\mathrm{LM}}{\mathrm{T}^2}\right]$."[31] To whom exactly these notations are due is not certain, as this edition of the reports was revised by several members of the committee. In any case, Maxwell used them in his famous Treatise of 1873. For example, [c] represents the dimensions of the current in the electrostatic system. [32] We'll do the same here, for practical reasons.

Maxwell and Jenkin defined, conceptually and operationally, all the electrical and magnetic quantities of any significance in use at the time: pole strength, magnetic field and moment, magnetization, induction coefficient, magnetic potential, quantity of electricity, resistance, etc. In most cases, they concluded with the dimensions of the quantity in question. In most cases, they conclude with the dimensions of the quantity in question. The dimensions obtained differ according to whether the electrostatic or electromagnetic system is used. As the committee had decided in favor of the latter, Maxwell and Jenkin concentrated on it, but they also dealt with the electrostatic system. They use capital letters to designate measurements of quantities in the electromagnetic system, and lower case letters for their measurements in the electrostatic system - a convention we shall follow

[29] B.A. Report 1873: 61.
[30] B.A. Report 1873: 70.
[31] "the *dimensions* of a force are $\left[\frac{\mathrm{LM}}{\mathrm{T}^2}\right]$"; B.A. Report 1873: 62.
[32] Maxwell 1873: vol.2, chapter 10, 241.

here.[33] At the end of their work, they indicate the dimensions of the most important physical quantities. They give the dimensions of magnetic quantities (pole intensity, magnet moment, magnetic field intensity) only in the electromagnetic system; but they give those of the main electrical quantities in both systems (quantity of electricity, current intensity, electromotive force and resistance).[34]

It's worth noting that, in the present context, Maxwell and Jenkin accord a virtually similar status to "units" and "dimensions." Indeed, when they summarize the results we have just derived, it is the dimensions of the quantities they indicate, in two summary lists whose titles refer to units: "electromagnetic system of *units*" and "electrostatic system of *units*."[35] Although the notion of a plurality of systems of units is due to Weber, it's worth noting that the fact that a physical quantity can have different dimensions in different systems seems to Maxwell and Jenkin just as natural as the fact that it can have different units. However, this was not self-evident; nothing in Fourier's treatment of dimensions suggested that such a situation was possible for a given group of fundamental quantities.

In both systems, the dimensions of the quantities are obtained from the same equations as the units, following the scheme already presented at the beginning of the report, and which Maxwell and Jenkin repeat in their appendix. It will be remembered that in Weber's electrostatic system, the units of the other quantities were obtained from that of the quantity of electricity, and the latter was defined on the basis of Coulomb's law. Maxwell and Jenkin did the same with dimensions. The dimensions of force f come from

[33] Maxwell & Fleeming 1863: 149.
[34] Maxwell & Fleeming 1863: 159.
[35] Maxwell & Fleeming 1863: 159.

Newton's second law, which gives . $\frac{\mathrm{ML}}{\mathrm{T}^2}$ [36] According to Coulomb's law $f = \frac{q_1\, q_2}{d^2}$, the dimensions of charge q are :[37]

$$\sqrt{\frac{\mathrm{M}\times\mathrm{L}}{\mathrm{T}^2}\times\mathrm{L}^2} = \frac{\mathrm{M}^{1/2}\mathrm{L}^{3/2}}{\mathrm{T}} \quad . \tag{33}$$

Note that these are fractional exponents.

In the electromagnetic system, Maxwell and Jenkin begin by calculating the dimensions of the current.[38] To do so, they consider the following device: a coil is suspended vertically, and a current is established in it, in a clockwise direction with respect to the observer. A bar magnet is suspended in the middle of the coil, and the magnet is seen to rotate so that it is aligned perpendicular to the coil axis, with its south pole facing the observer:

Sou

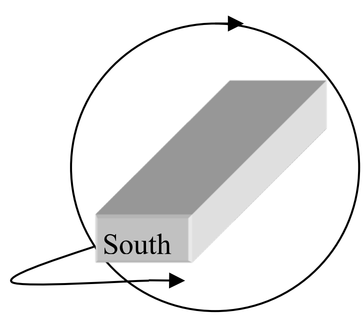

Figure 3.1

[36] Maxwell & Fleeming 1863: 148-149.
[37] See also Maxwell 1873: vol.1, section "Dimensions of the Electrostatic Unit of Quantity," 43.
[38] Maxwell & Fleeming 1863: 139-140.

Maxwell and Jenkin used the equation that relates the moment of force exerted by the current on the bar magnet to the current, in order to define its unit - and dimensions. Thus, we have :

$$G = mlC.2\pi nk\frac{1}{k^2}\cos\theta, \quad (34)$$

where : G is the moment of the force
ml: the "magnet moment
C: current
n: number of turns in the coil
k: radius of the circle
θ the angle between the axes of the circle and the magnet.[39]

From this relationship, Maxwell and Jenkin defined the unit of current as the quantity of current required, in a unit length of electric wire, to produce a unit of magnetic force at the center of a coil of a unit length of radius - in other words, the magnet must carry a unit of magnetic pole.

For the dimensions, they consider the force exerted by the current on one pole of the magnet (rather than the moment of the force on the magnet) : [40]

$$f = \frac{mCL}{L_1^2}, \quad (35)$$

where L_1 is simply the radius of the circle (represented by k in the previous formula) and L is the length of the wire (corresponding to). $2\pi nk$

This gives :

$$C = \frac{L_1^2 f}{mL} \quad (36)$$

[39] Maxwell & Fleeming 1863: 139.
[40] Maxwell & Fleeming 1863: 140.

Of course, f has dimensions MLT^{-2} , and the authors have already assigned the magnetic pole m dimensions $\mathrm{L}^{3/2}\,\mathrm{M}^{1/2}\,\mathrm{T}^{-1}$, from the equation $f=\frac{m_1 m_2}{d^2}$. Therefore, the dimensions of m are $\left(\mathrm{MLT}^{-2}\right)^{1/2}\mathrm{L}=\mathrm{L}^{3/2}\,\mathrm{M}^{1/2}\,\mathrm{T}^{-1}$. Consequently, the electric current has dimensions :[41]

$$[C]=\frac{\mathrm{L}^2\times\mathrm{MLT}^{-2}}{\mathrm{L}^{3/2}\mathrm{M}^{1/2}\,\mathrm{T}^{-1}\times\mathrm{L}}=\frac{\mathrm{L}^{1/2}\mathrm{M}^{1/2}}{\mathrm{T}}. \tag{37}$$

Obtaining the dimensions of electric charge from this result is straightforward: simply use the definition of electric current as the quantity of charge passing per unit time. The dimensions of the charge are therefore those of the current multiplied by T: [42]

$$[Q]=\mathrm{L}^{1/2}\,\mathrm{M}^{1/2}\ . \tag{38}$$

Maxwell and Jenkin also derived formulas for the dimensions of other quantities in both systems. We've just seen how they derive the dimensions of current in the electromagnetic system. In the electrostatic system, they derive it from the dimensions of charge:

$$[c]=\frac{[q]}{\mathrm{T}}=\frac{\mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-1}}{\mathrm{T}}=\frac{\mathrm{L}^{3/2}\,\mathrm{M}^{1/2}}{\mathrm{T}^2}, \tag{39}$$

where the lower-case letters c and q refer to the electrostatic system.

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[41] Maxwell obtained a similar result in his treatise, but this time considering the force between two current elements: $F=2ii'\frac{s}{r}$ where r is the distance separating the conductors, and s the length of the conductor considered. From a dimensional point of view, this equation implies :

$$[F]=[i^2]\cdot\left[\frac{\mathrm{L}}{\mathrm{L}}\right]$$

hence : $[i]=[F]^{1/2}$ i.e. . $[i]=[\mathrm{MLT}^{-2}]^{1/2}=\mathrm{M}^{1/2}\mathrm{L}^{1/2}\mathrm{T}^{-1}$
Maxwell 1873: vol.2, 141.

[42] Maxwell & Fleeming 1863: 144.

They also derive dimensional formulas for resistance and electromotive force in both systems.[43] They do not always use exactly the same equations as those given by the Committee in the first part of its report. For example, to obtain the dimensions of the electromotive force, they use the fact that the work done on a load is equal to the quantity of load on which it is exerted multiplied by the electromotive force: [44]

$$W = EQ\,, \tag{40}$$

where W represents work, E the "electromotive force" and Q the quantity of charge. Since the dimensions of work are those of a force multiplied by a distance, i.e. $\mathrm{ML^2T^{-2}}$, we obtain :

$$[E] = \frac{[W]}{[Q]} = \frac{\mathrm{ML^2T^{-2}}}{[Q]}\,. \tag{41}$$

This method enables them to obtain the dimensions of the electromotive force not only in the electromagnetic system, but also in the electrostatic system. In the former, the above result implies, if we use the dimensions obtained for the charge from electromagnetic phenomena :

$$[E] = \frac{\mathrm{ML^2T^{-2}}}{\mathrm{L^{1/2}M^{1/2}}} = \frac{\mathrm{L^{3/2}\,M^{1/2}}}{\mathrm{T^2}}\,; \tag{42}$$

and in the electrostatic system, using the dimensions of the quantity of electricity in the latter, we obtain: [45]

$$[e] = \frac{\mathrm{ML^2T^{-2}}}{\mathrm{M^{1/2}L^{3/2}T^{-1}}} = \frac{\mathrm{L^{1/2}\,M^{1/2}}}{\mathrm{T}}\,. \tag{43}$$

[43] Maxwell & Fleeming 1863: 151.
[44] Maxwell & Fleeming 1863: 144-145.
[45] Maxwell & Fleeming 1863: 150.

To obtain the dimensions of the resistance, Maxwell and Jenkin use equation (1) given at the beginning of the report, . $C = \frac{E}{R}$ [46] It follows that the dimensions of electrical resistance in the electrostatic case are :

$$[r] = \frac{[e]}{[c]} = \frac{\mathrm{M}^{1/2}\mathrm{L}^{1/2}\mathrm{T}^{-1}}{\mathrm{M}^{1/2}\,\mathrm{L}^{3/2}\mathrm{T}^{-2}} = \frac{\mathrm{T}}{\mathrm{L}}. \qquad \textbf{(44)}$$

and in the electromagnetic case :[47]

$$[R] = \frac{[E]}{[C]} = \frac{\mathrm{M}^{1/2}\,\mathrm{L}^{3/2}\,\mathrm{T}^{-2}}{\mathrm{M}^{1/2}\,\mathrm{L}^{1/2}\,\mathrm{T}^{-1}} = \frac{\mathrm{L}}{\mathrm{T}}. \qquad \textbf{(45)}$$

Here we return to Weber's result that the dimensions of resistance are those of velocity.

But Maxwell and Jenkin don't just describe the two systems. They examine their mutual relationship, and also the influence that a material medium can have on them. Their interest in this question is not purely theoretical - although its importance from this point of view is fundamental. The ability to switch from the electromagnetic to the electrostatic system was essential for telegraph engineers. Thomson demonstrated in 1854 that current transmission in telegraph cables depends on RCl^2 where R is the cable resistance, C its capacitance and l its length. More precisely, the time required for a given fraction of the maximum current to reach the end of the cable is proportional to this quantity.[48] Capacitance measurements were electrostatic, while resistance measurements were electromagnetic. Despite the Committee's preference, electrostatic measurements retained

[46] Maxwell & Fleeming 1863: 145.
[47] Maxwell & Fleeming 1863: 145.
[48] Thomson W. to Stokes, October 28, 1854.

their practical importance. The ability to convert between the two systems was therefore essential for telegraph signal damping calculations.[49]

Maxwell and Jenkin began by insisting that the relationships between the various physical quantities - i.e. the equations - remain valid whatever the system used:

> The relationships between current and quantity [of electricity], between work, current, and electromotive force, and between electromotive force, current and resistance, are not affected if we switch from the electromagnetic to the electrostatic system.[50]

This may seem obvious, but it is nonetheless in line with Fourier's concerns. Maxwell and Jenkin then establish a relationship between the expression of quantities in each of the two systems, starting with the charge. To do this, they consider the dimensions of the quotient $\frac{q}{Q}$. Having previously obtained $\frac{\mathrm{M}^{1/2}\mathrm{L}^{3/2}}{\mathrm{T}}$ for the dimensions of q and $\mathrm{L}^{1/2}\ \mathrm{M}^{1/2}$ for those of Q, they obtain for the dimensions of : $\frac{q}{Q}$ [51]

$$\left[\frac{q}{Q}\right] = \frac{\mathrm{M}^{1/2}\ \mathrm{L}^{3/2}\ \mathrm{T}^{-1}}{\mathrm{L}^{1/2}\mathrm{M}^{1/2}} = \mathrm{L}^{3/2-1/2}\mathrm{T}^{-1} = \mathrm{LT}^{-1}. \quad \textbf{(46)}$$

therefore the dimensions of a velocity, which they represent by *v.*

This is in line with what Maxwell will indicate in his *Treatise*, but seems to contradict what Weber obtained, and differs from what they present further on. Indeed, below Maxwell and Jenkin state: "The electromagnetic unit of quantity of electricity is

[49] Darrigol 2000: 124-125. See also Wise & Smith 1987: 329.
[50] "The relations between current and quantity, between work, current, and electromotive force, and between electromotive force, current, and resistance, remain unchanged by the change from the electromagnetic to the electrostatic system." Maxwell & Fleeming 1863: 149.
[51] Maxwell & Fleeming 1863: 149.

equal to the electrostatic unit multiplied by a certain velocity"[52] and from this obtain the dimensions of the electrostatic quantity by dividing those of the electromagnetic quantity by LT^{-1} . This seems to be due to confusion, itself linked to a difference in perspective between Maxwell and Jenkin on the one hand, and Weber on the other. To clarify, in what follows we systematically use square brackets to indicate the concrete units of quantities, and symbols without brackets for their measurements.[53]

In all cases, it is understood that the electrostatic measure of charge is *v* times the electromagnetic measure: $q = vQ$ (as is necessary for Coulomb's and Ampère's laws, respectively, to be written without coefficients). This implies that, for a given choice of units of time and length, the electromagnetic unit of charge *corresponds* to *v* electrostatic units: when , . $Q = 1$ $q = v$

For Weber, there is only one physical quantity corresponding to electric charge, so electrostatic and electromagnetic measurements differ because the corresponding units differ: $Q[Q] = q[q]$. With $q = vQ$, this implies $[Q] = v[q]$, which might seem at odds with the result obtained above by the two Britons.

In fact, Maxwell and Jenkin have an entirely different perspective. They refer to the quotient $\frac{q}{Q}$ itself as velocity (by which they obviously mean $\left[\frac{q}{Q}\right]$). This implies that they consider charge *q* and charge *Q* as two different, albeit proportional, physical quantities, the coefficient of proportionality being a velocity:

$$q[q] = v\frac{\mathrm{L}}{\mathrm{T}}Q[Q] \qquad \textbf{(47)}$$

[52] "The electromagnetic unit of quantity of electricity is equal to the electrostatic unit multiplied by a certain velocity" Maxwell & Fleeming 1863: 161.
[53] In the same way that d [d] could represent100 km where d corresponds to 100 and [d] to .1 km

In this case, for the electrostatic measure of charge to be v times the electromagnetic measure ($q = vQ$), the unit of electrostatic charge must be $\frac{\mathrm{L}}{\mathrm{T}}$ times the electromagnetic unit : $[q] = \frac{\mathrm{L}}{\mathrm{T}}[Q]$ as they obtain above. Nevertheless, an electromagnetic charge unit ($Q = 1$ *) corresponds to v* electrostatic charge units, as Weber pointed out, since $q = v$. This seems to be what led them to make a misinterpretation later on in their report: they then seem to translate this fact as $[Q] = v[q]$ (which would correspond to Weber's conceptions but makes no sense in theirs), and deduce from this $[Q] = \frac{\mathrm{L}}{\mathrm{T}}[q]$, in direct contradiction with the result $[q] = \frac{\mathrm{L}}{\mathrm{T}}[Q]$ (neither of these last two expressions making sense from Weber's point of view). Thus, Maxwell and Jenkin seem to have confused the ratio of the units of the two types of charge with the number of units of one type of charge corresponding to one unit of the other type. It would appear that they later realized this, as the section of their article in which the confusion takes place is absent from the second edition of 1873.

Maxwell and Jenkin then present similar relationships between expressions in the two systems of units for the three other quantities they dealt with.[54] Thus:

$$\left[\frac{c}{C}\right] = \frac{\mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-2}}{\mathrm{L}^{1/2}\mathrm{M}^{1/2}\mathrm{T}^{-1}} = \mathrm{LT}^{-1} \qquad \left[\frac{e}{E}\right] = \frac{\mathrm{M}^{1/2}\mathrm{L}^{1/2}\mathrm{T}^{-1}}{\mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-2}} = \mathrm{L}^{-1}\mathrm{T} \qquad \left[\frac{r}{R}\right] = \frac{\mathrm{L}^{-1}\,\mathrm{T}}{\mathrm{LT}^{-1}} = \mathrm{L}^{-2}\mathrm{T}^{2} \qquad \textbf{(48)}$$

and therefore :

$$c = vC, \qquad e = \frac{E}{v}, \qquad r = \frac{R}{v^2}. \qquad \textbf{(49)}$$

[54] Maxwell & Fleeming 1863: 150.

For Maxwell, the speed *v* plays a fundamental role, since in his 1861-62 theory it gives the speed of wave propagation in the medium carrying electromagnetic actions, waves he identifies with light.[55] In 1863, Maxwell and Jenkin mentioned that Weber and Kohlrausch had determined this speed experimentally and obtained a value very close to the speed of light:

> 310,740,000 meters per second - a speed that differs less from the estimated speed of light than the different estimates of the latter.[56]

They attribute the first estimate of the value of this speed to Faraday.

Maxwell and Jenkin then examined the influence of the medium. This led them to define two other systems of units, relevant when dealing with phenomena taking place in media other than air. Their motivations are worth examining in detail:

> All the measurements discussed so far are assumed to have been made in the same medium - ordinary air; but Faraday has shown that other media have different properties. Paramagnetic bodies, such as oxygen and iron salts, when placed in a medium less paramagnetic than themselves, behave as paramagnetic bodies; but when placed in a medium more paramagnetic than themselves, they behave as diamagnetic bodies.
> Magnetic phenomena are therefore influenced by the nature of the medium in which the bodies are placed, and the system of units and measurements we adopt depends on the nature of the medium in which our experiments are carried out. If we carried out our experiments in highly condensed oxygen, the magnets would attract each other less, and the currents would attract each other more, than they do in air; and the opposite would be true if we worked in a sea of liquid bismuth.[57]

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[55] Maxwell 1861a: 161-175, 281-291, 338-348, Maxwell 1861b: 12-24, 85-95.
See also the following later publications: Maxwell 1865: 459; reported in Maxwell 1868: 652; and also Maxwell 1868: 643-657, article devoted to the experimental determination of *v* - followed by a note on the electromagnetic theory of light.

[56] "310, 740,000 metres per second - a velocity not differing from the estimated velocity of light by more than the different determinations of the latter quantity differ from each other," Maxwell & Fleeming 1863: 149.

[57] "All the measurements of which we have hitherto treated are supposed to be made in the same medium - ordinary air; but Faraday has shown that other media have different properties. Paramagnetic bodies, such as

What may seem surprising is that, according to Maxwell and Jenkin, taking into account the influence of the environment requires modifying the dimensions of the magnetic pole - hence the need for them to define two new systems:

> If we now take into account the "magnetic induction coefficient" of the medium in which we're working, and instead of taking that of air as the unit, assume that it's proportional to the density of the part of the medium to which the magnetic action is due, we'll have for the repulsion between the two poles $= \frac{m m^1}{\mu r^2}$ where mm^1 are the two poles, μ the density of the magnetic medium, and r the distance. Now, a density is a mass, M, divided by $L^{(3)}$ th ^{e}unit of volume. Therefore, m has dimensions $\sqrt{\frac{MM_1}{T^2}}$; or, if we can measure the density of the magnetic medium in the same unit of mass that is used for other purposes, m would simply have dimensions . $\frac{M}{T}$ [58]

Since the electromagnetic system is based on the dimensions of the magnetic pole, the dimensions of all quantities expressed in this system are affected. For current, we had :

$$[C] = \left[\frac{L_1{}^2 f}{m\mathrm{L}}\right], \qquad \textbf{(50)}$$

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oxygen and salts of iron, when placed in media less paramagnetic than themselves, behave as paramagnetic bodies; but when placed in media more paramagnetic than themselves, they behave as diamagnetic bodies. Hence magnetic phenomena are influenced by the nature of the medium in which the bodies are placed, and the system of units and of measurements which we adopt depends on the nature of the medium in which our experiments are made. If we made our experiments in highly condensed oxygen, magnets would attract each other less, and currents would attract each other more, than they do in common air; and the reverse would be the case if we worked in a sea of melted bismuth." Maxwell & Fleeming 1863: 160.

[58] "Now if we take into account the "coefficient of magnetic induction "of the medium in which we work, and instead of assuming that of common air to be unity, assume it proportional to the density of that part of the medium to which the magnetic action is due, we shall have the repulsion of two poles $= \frac{m m^1}{\mu r^2}$, where $mm^1$are the two poles, μ the density of the magnetic medium, and r the distance. Now a density is a mass, M, divided by L^3, the unit of volume. Hence the dimensions of m are $\sqrt{\frac{MM_1}{T^2}}$; or, if we can measure the density of the magnetic medium in the same unit of mass as that employed for other purposes, the dimensions of m would be simply $\frac{M}{T}$." Maxwell & Fleeming 1863: 160-161.

which, with $[m] = \mathrm{L}^{3/2}\mathrm{M}^{1/2}\,\mathrm{T}^{-1}$, implied :

$$[C] = \frac{\mathrm{L}^2 \cdot \mathrm{MLT}^{-2}}{\mathrm{L}^{3/2}\mathrm{M}^{1/2}\,\mathrm{T}^{-1} \cdot \mathrm{L}} = \frac{\mathrm{L}^{1/2}\,\mathrm{M}^{1/2}}{\mathrm{T}}\ . \tag{51}$$

We now have : $[m] = \mathrm{M\,T}^{-1}$

$$[C] = \frac{\mathrm{L}^2 \cdot \mathrm{MLT}^{-2}}{\mathrm{MT}^{-1} \cdot \mathrm{L}} = \frac{\mathrm{L}^2}{\mathrm{T}} \tag{52}$$

and therefore, for the electric charge *Q*, instead of : $\mathrm{L}^{1/2}\mathrm{M}^{1/2}$

$$[C] \cdot \mathrm{T} = \mathrm{L}^2\mathrm{T}^{-1} \cdot \mathrm{T} = \mathrm{L}^2 \tag{53}$$

for the electromotive force, instead of $\mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-2}$:

$$[E] = \frac{[W]}{[Q]} = \frac{\mathrm{ML}^2\mathrm{T}^{-2}}{[Q]} = \frac{\mathrm{ML}^2\mathrm{T}^{-2}}{\mathrm{L}^2} = \frac{\mathrm{M}}{\mathrm{T}^2} \tag{54}$$

and for resistance, instead of : LT^{-1}

$$[R] = \frac{[E]}{[C]} = \frac{\mathrm{MT}^{-2}}{\mathrm{L}^2\mathrm{T}^{-1}} = \frac{\mathrm{M}}{\mathrm{L}^2\,\mathrm{T}}\,. \tag{55}$$

A priori, one might expect the electrostatic system to be unaffected; instead, Maxwell and Jenkin derive a fourth system of units. They assume that

> The electromagnetic unit of electrical quantity is equal to the electrostatic unit multiplied by a certain speed, depending on the elasticity of the magnetic medium, and proportional or probably equal to the speed of propagation of vibrations in this medium.[59]

For the dimensions of the electric charge, this gives :

$$[q] = \frac{[Q]}{[V]} = \frac{\mathrm{L}^2}{\mathrm{LT}^{-1}} = \mathrm{LT} \tag{56}$$

[59] "The electromagnetic unit of quantity of electricity is equal to the electrostatic unit multiplied by a certain velocity, depending on the elasticity of the magnetic medium, and proportional or probably equal to the velocity of propagation of vibrations in it." Maxwell & Fleeming 1863: 161.

and, for other quantities: [60]

$$[c] = \frac{[q]}{\mathrm{T}} = \frac{\mathrm{LT}}{\mathrm{T}} = \mathrm{L}\,, \tag{57}$$

$$[e] = \frac{[w]}{[q]} = \frac{\mathrm{ML^2T^{-2}}}{\mathrm{LT}} = \frac{\mathrm{ML}}{\mathrm{T^3}}\,, \tag{58}$$

$$[r] = \frac{[e]}{[c]} = \frac{\mathrm{MLT^{-3}}}{\mathrm{L}} = \frac{\mathrm{M}}{\mathrm{T^3}}\,. \tag{59}$$

The question of whether or not to take the environment into account, and the consequences of this for the dimensions of the relevant quantities, will be the subject of much debate in the rest of the century. It is therefore interesting to note that it already appears in the present work. Overall, Maxwell and Jenkin's paper reveals the complexity of the dimensional question much better than Fourier's earlier work. It shows that dimensions depend not only on the phenomena and equations one chooses to base them on (electrostatic or electrodynamic system), but also on one's conception of the phenomena involved (intervention of a medium or not).

While the Committee's work is heir to that of Gauss and Weber in its mechanistic conceptions, Maxwell and Jenkin's approach is also inspired by that of Fourier. In the early 1860s in Great Britain, a synthesis was achieved between the contributions of Gauss and Weber on the one hand, and Fourier on the other. The impact of the Committee's 1863 report cannot be overstated. Its decisions were adopted first in Great Britain, before

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[60] Maxwell & Fleeming 1863: 161.
On the other hand, if we use $\frac{[q]}{[Q]} = [v] = \mathrm{LT^{-1}}$, we obtain :

$$[c] = \frac{[q]}{\mathrm{T}} = \frac{\mathrm{L^3T^{-1}}}{\mathrm{T}} = \frac{\mathrm{L^3}}{\mathrm{T^2}} \qquad [e] = \frac{[w]}{[q]} = \frac{\mathrm{ML^2T^{-2}}}{\mathrm{L^3T^{-1}}} = \frac{\mathrm{M}}{\mathrm{LT}} \qquad [r] = \frac{[e]}{[c]} = \frac{\mathrm{ML^{-1}T^{-1}}}{\mathrm{L^3T^{-2}}} = \frac{\mathrm{MT}}{\mathrm{L^4}}$$

gradually spreading across the continent. Ironically, it is the unity of resistance that will remain the most problematic. In France and other countries, this impact was largely due to Maxwell's later work.

3. 3 - Dimensions and the vector nature of quantities: Maxwell's 1871 paper

In 1871, Maxwell briefly used the concept of dimension as part of a discussion of the classification of physical quantities.[61] He distinguished between scalar and vector quantities, and also differentiated between different types of vectors within the latter. It is in this context that original dimensional considerations come into play.[62] Maxwell uses the dimensions of energy, $\frac{ML^2}{T^2}$, to list the factors into which it can be broken down. He explains that energy is a scalar quantity, and that if we divide energy into two factors, one of which contains L^2, these two factors are also scalar quantities (he takes the example of mass times velocity squared, as in kinetic energy). If, on the other hand, we divide it into two factors, each of which contains L, these factors are both vectors (he then mentions force and displacement, or momentum and velocity). We can deduce from this, even if Maxwell doesn't say so explicitly, that the dimension formula of a physical quantity allows us to determine its scalar or vector nature as a function of the dimension exponent of L: if the exponent is odd, the quantity is vector, if it is even (including 0), it is scalar. These considerations are quite different from those in the 1863 article. On the other hand, the

[61] Maxwell 1871: 224-233.
[62] Maxwell 1871: 226-227.

much more developed discussion from the point of view of dimensional analysis that he offers in 1873 is again in the spirit of the latter.

3. 4 - Maxwell's treatise

In his famous *Treatise on Electricity and Magnetism* (1873), Maxwell developed new ideas on dimensions in general and electrical quantities in particular. His motivations were the same as his earlier concerns:

> In all scientific studies, it is of the utmost importance to use units belonging to well-defined systems, and to know the relationships between these units and the fundamental units, so that we can immediately transform our results from one system to another.[63]

Maxwell first clarifies several important points concerning one of the fundamental quantities, mass.[64] He explains that there are at least two different ways of defining the unit of mass: one using the metric system, the other based on astronomical considerations. In the metric system, the gram is defined as the mass of one cubic centimeter of water under normal conditions of temperature and pressure. (To this theoretical definition is added a "practical consideration", namely that the gram is taken as one thousandth of the standard kilogram kept in Paris).[65] In astronomy, the unit of mass is defined as the mass that attracts another body located one unit away in such a way that this second body is accelerated by

[63] "In all scientific studies it is of the greatest importance to employ units belonging to a properly defined system, and to know the relations of these units to the fundamental units, so that we may be able at once to transform our results from one system to another." Maxwell 1873: vol.1, 2.
[64] In it, Maxwell simply mentions that the unit of mass in Great Britain is the avoirdupoids pound, that the "grain" is one seven-thousandth of this, and that in France the gram "derived from the unit of length, by the use of water at a standard temperature as a standard of density" is used. Maxwell & Fleeming 1863: 131-132.
[65] Maxwell 1873: vol.1, 3.

one unit of acceleration. (In addition, Maxwell specifies that the mass of the sun or the Earth are sometimes taken as units of mass). [66]

It's worth noting - and Maxwell doesn't - that while we're clearly dealing with gravitational mass in the second case, the situation isn't so clear-cut in the first: given that we're referring to a quantity of matter defined by a volume, we're dealing directly neither with inertial mass (which we'd obtain from measurements of force and acceleration) nor with gravitational mass (which we'd obtain, for example, from a measurement of weight), but with a quantity that can only be said to be proportional to them.

The advantages of these two different theoretical definitions are twofold. Firstly, each allows mass to be expressed in terms of the other two fundamental quantities, length and time. This raises the question of the extent to which mass can therefore be considered truly fundamental - a point not disputed by Maxwell. Secondly, the two definitions lead to two dimensional formulas for mass, in terms of L and T, which happen to differ from one another.[67] In fact, in the metric system, since the density of water is taken to be standard and therefore constant, mass is proportional to volume. For dimensions, Maxwell concludes:[68]

$$M = L^3 . \qquad \textbf{(60)}$$

[66] Maxwell 1873: vol.1, 3-4.
[67] Maxwell 1873: vol.1, 4.
[68] Maxwell does not explicitly write this equation. However, he explicitly states that in the astronomical system, the dimensions of M are L^3T^{-2} ; Maxwell 1873: vol.1, 4.

On the other hand, in the astronomical system, mass is proportional to acceleration and proportional to the square of length.[69] We therefore have :[70]

$$\mathrm{M} = \mathrm{LT}^{-2} \cdot \mathrm{L}^2 = \mathrm{L}^3\mathrm{T}^{-2} \tag{61}$$

Maxwell makes no comment on these results. It would seem, then, that he found nothing surprising in the fact that a mechanical quantity such as mass could have several different dimensional formulas for the same choice of fundamental quantities. He was certainly accustomed to the fact that the same physical quantity could have several dimensional formulas, due to the duality of the electrostatic and electromagnetic systems.

Perhaps the most interesting aspect of Maxwell's treatise for our purposes is the discussion he offers in a chapter of his second volume, entitled "Dimensions of electric units."[71] In it, Maxwell determines relationships between the dimensions of electrical (and magnetic) quantities that are independent of the unit system considered. This suggests that Maxwell wanted to go further in his dimensional considerations than determining the dimensions of quantities in different systems and the relationship between them. Here, he seems to be attempting to develop more fundamental relationships, which he then used to deduce the electrostatic and electromagnetic systems.

Maxwell began by defining the various electrical and magnetic quantities, classifying them in such a way that it was easy for him to define certain dimensional relationships between them, independent of the system used. He thus describes several

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[69] Acceleration of the second mass = $\dfrac{\text{masse unité}}{(\text{distance entre les masses})^2}$
hence unit mass = acceleration × distance^2.

[70] In fact, Maxwell does not give himself the dimensions of an acceleration directly, but uses the formula $s = \dfrac{1}{2}at^2$ in which he replaces a by $\dfrac{m}{r^2}$, then solves $s = \dfrac{1}{2}\dfrac{m}{r^2}t^2$ for *m*, obtaining: $m = \dfrac{2sr^2}{t^2}$

[71] Maxwell 1873: vol.2, chapter 10, 239-245.

pairs of quantities, whose product has the same dimensions in both systems. In all, he gives six pairs of quantities. For the first three, the dimensions of the product of each pair are those of energy or work. For the last three, the dimensions of the product are those of energy density - that is, energy divided by volume.[72] Maxwell's interest in these quantities parallels the importance of the notion of energy in Great Britain's choice of unit systems.[73]

These pairs of quantities, and their corresponding symbols, are as follows: [74]

Pairs whose product has energy or work dimensions	Pairs whose product has the dimensions of energy per unit volume
Electrostatic pair :	Electrostatic pair :
- Quantity of electricity e - Integral of the electromotive force along a line or electric potential E	- Electrical displacement (measured by surface density) : $\mathcal{D}$ - Electromotive force at a point[75] $\mathcal{F}$
Magnetic pair :	Magnetic pair :
- Quantity of free magnetism, or force of one pole: m - Magnetic potential Ω	- Magnetic induction $\mathcal{B}$ - Magnetic force[52] $\mathcal{H}$
Electrokinetic pair :	Electrokinetic pair :
- Electrokinetic momentum of a circuit: p - Electrical current: C	- Electric current at a point : $\mathcal{C}$ - Potential vector of electric currents : $\mathcal{U}$

Table 3.1

[72] Maxwell 1873: vol.2, chapter 10, 239-240.
[73] Smith 1998.
[74] Here, we follow the French translation of Maxwell's treatise by G. Séligmann-Lui, a telegraph engineer, published by Gauthier-Villars in 1883, volume 2, 295-296. We present these results in table form for greater clarity, but Maxwell presents them simply as a list.
[75] These forces correspond to what we now call fields (Maxwell considered them as acting on a charge or a unit pole).

Given that the dimensions of an energy are $\frac{\mathrm{L}^2\mathrm{M}}{\mathrm{T}^2}$ and those of an energy per unit volume $\frac{\mathrm{M}}{\mathrm{LT}^2}$, we obtain the following dimensional relationships - which, once again, are independent of the system under consideration:

$$[eE]=[m\Omega]=[pC]=\frac{\mathrm{L}^2\mathrm{M}}{\mathrm{T}^2} \tag{62}$$

$$[\mathcal{D}\mathcal{F}]=[\mathcal{B}\mathcal{H}]=[\mathcal{C}\,\mathcal{U}]=\frac{\mathrm{M}}{\mathrm{LT}^2} \tag{63}$$

Maxwell gives nine further dimensional relations that are independent of the system under consideration. He takes the quotients of the quantities defined in the table above, when one of these quantities is an integral of another. Thus, since *e*, *p* and $\mathcal{U}$ are the respective integrals of *C*, *E* and $\mathcal{F}$ with respect to time:[76]

$$\left[\frac{e}{C}\right]=\left[\frac{p}{E}\right]=\left[\frac{\mathcal{U}}{\mathcal{F}}\right]=\mathrm{T} \tag{64}$$

Since *E*, *Ω* and *p* are the integrals of ,$\mathcal{F}$ $\mathcal{H}$ and $\mathcal{U}$ along a line :

$$\left[\frac{E}{\mathcal{F}}\right]=\left[\frac{\Omega}{\mathcal{H}}\right]=\left[\frac{p}{\mathcal{U}}\right]=\mathrm{L} \tag{65}$$

And since *e*, *C* and *m* are the surface integrals of $\mathcal{D}$, $\mathcal{C}$ and $\mathcal{B}$:

$$\left[\frac{e}{\mathcal{D}}\right]=\left[\frac{C}{\mathcal{C}}\right]=\left[\frac{m}{\mathcal{B}}\right]=\mathrm{L}^2 \tag{66}$$

By taking the products or quotients of the quantities defined in the table, Maxwell obtains a total of fifteen dimensional equations independent of the system of units considered (three per line). At the same time, we have twelve electrical or magnetic quantities (since we have six pairs of quantities). However, the fifteen equations are not independent, and they do

[76] Maxwell 1873: vol.2, chapter 10, 240.

not allow us to determine all the quantities. In fact, to determine all twelve quantities, we'd need one more equation. [77]

What Maxwell proposes is to eliminate an unknown. He decided to treat either e or m in the same way as the fundamental mechanical quantities. He therefore expressed the dimensions of each of the twelve quantities in terms of, firstly, M, L, T, e, and secondly, M, L, T, m.[78] This, of course, enabled him to define the electrostatic and electromagnetic systems, but it must be stressed that, at this stage, the dimensional relations he obtained are not strictly speaking system-specific, since they are not formulas involving only L, M and T: they can be said to be true in the absolute, and not just in a particular system.

Maxwell thus obtains the following dimensional relations:

$$[e] = \mathrm{e} = \frac{\mathrm{L}^2\,\mathrm{M}}{\mathrm{m\,T}} \tag{67}$$

$$[E] = \frac{\mathrm{L}^2\,\mathrm{M}}{\mathrm{e\,T}^2} = \frac{\mathrm{m}}{\mathrm{T}} \tag{68}$$

$$[p] = [m] = \frac{\mathrm{L}^2\,\mathrm{M}}{\mathrm{e\,T}} = \mathrm{m} \tag{69}$$

$$[C] = [\Omega] = \frac{\mathrm{e}}{\mathrm{T}} = \frac{\mathrm{L}^2\,\mathrm{M}}{\mathrm{m\,T}^2} \tag{70}$$

$$[\mathcal{D}] = \frac{\mathrm{e}}{\mathrm{L}^2} = \frac{\mathrm{M}}{\mathrm{m\,T}} \tag{71}$$

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[77] Firstly, $[eE]=[pC]$ appears in both (36) and (38).

Secondly, from (37) we have from which $[\mathcal{D}\mathcal{F}]=[C\mathcal{U}]$ $\left[\frac{\mathcal{U}}{\mathcal{F}}\right]=\left[\frac{\mathcal{D}}{C}\right]$ and from (40) $\left[\frac{e}{\mathcal{D}}\right]=\left[\frac{C}{C}\right]$ from which $\left[\frac{\mathcal{D}}{C}\right]=\left[\frac{e}{C}\right]$, so $\left[\frac{\mathcal{U}}{\mathcal{F}}\right]=\left[\frac{e}{C}\right]$, which we already know from (38). Thirdly, we know:

$\left[\frac{m}{e}\right]\overset{(40)}{=}\left[\frac{\mathcal{B}}{\mathcal{D}}\right]\overset{(37)}{=}\left[\frac{\mathcal{F}}{\mathcal{H}}\right]\overset{(39)}{=}\left[\frac{E}{\Omega}\right]\left[\frac{m}{e}\right]=\left[\frac{E}{\Omega}\right]$ by (36).

Fourth, we have:

$\left[\frac{\mathcal{B}}{\mathcal{D}}\right]\overset{(37)}{=}\left[\frac{\mathcal{F}}{\mathcal{H}}\right]\overset{(39)}{=}\left[\frac{E}{\Omega}\right]\overset{(36)}{=}\left[\frac{m}{e}\right]$ but we already know $\left[\frac{\mathcal{B}}{\mathcal{D}}\right]=\left[\frac{m}{e}\right]$ from (40).

[78] Maxwell 1873: vol.2, chapter 10, 241.

$$[\mathcal{F}] = \frac{L\,M}{e\,T^2} = \frac{m}{L\,T} \qquad \textbf{(72)}$$

$$[\mathcal{B}] = \frac{M}{e\,T} = \frac{m}{L^2} \qquad \textbf{(73)}$$

$$[\mathcal{H}] = \frac{e}{L\,T} = \frac{L\,M}{m\,T^2} \qquad \textbf{(74)}$$

$$[\mathcal{C}] = \frac{e}{L^2\,T} = \frac{M}{m\,T^2} \qquad \textbf{(75)}$$

$$[\mathcal{U}] = \frac{L\,M}{e\,T} = \frac{m}{L} \qquad \textbf{(76)}$$

Maxwell underlines the similarity in structure between formulas containing e and those containing m for ten of the quantities. He ranks them as follows:

As a function of e :

e $\mathcal{D}$ $\mathcal{H}$ C & Ω E $\mathcal{F}$ $\mathcal{B}$ m and p

As a function of m :

m and p $\mathcal{B}$ $\mathcal{F}$ E $\mathcal{C}$ & Ω $\mathcal{H}$ $\mathcal{D}$ e

The dimensions as a function of e of the quantities in the first line have the same shape as the dimensions as a function of m of the corresponding quantities in the second line. For example, $\mathcal{D}$ has dimensions as a function of e $\frac{e}{L^2}$, and $\mathcal{B}$ has dimensions as a function of m $\frac{m}{L^2}$. In addition to this one-to-one relationship between the quantities in the two lines, Maxwell notes that the quantities are arranged in reverse order in both lists, and that the four quantities on the left for each line have either e or m in the numerator of their dimension formula, while those on the right have it in the denominator. Curiously, he draws no conclusions from these structural similarities. [79]

Maxwell then proposes to derive the electrostatic and electromagnetic systems from the dimensional formulas given above. It must be emphasized that it is not a question here of deriving the electrostatic system from the series of formulas in terms of M, L, T and e, and the electromagnetic system from those in terms of M, L, T and m. On the contrary, each of the two systems can be obtained from both series of formulas, since the equivalence of the two systems is clear. On the contrary, each of the two systems can be obtained from both sets of formulas, since the equivalence of these two sets is, once again, independent of the system under consideration. What the problem consists in is determining the dimensions of e and m in each of the two systems as a function of M, L and T only, so as to be able to substitute the dimension formulas in question - either for e or for m - into the formulas already obtained. Despite the originality of the preceding discussion, the approach Maxwell now follows is therefore quite similar to that of the 1863 article, whose results he takes up (already discussed elsewhere in the *Treatise*). He obtains the dimensions of e and m in the electrostatic system on the one hand, and in the electromagnetic system on the other. Only, instead of starting from Coulomb's and Ampère's laws, Maxwell starts here from the corresponding field laws, since fields are part of the quantities he has just defined. His electrostatic system is thus based on the relation :

$$\mathcal{F} = \frac{e}{L^2}, \qquad \textbf{(77)}$$

[79] Maxwell 1873: vol.2, chapter 10, 241.

where $\mathcal{F}$, described above as the "electromotive force at a point" (our "electric field"), corresponds to the electrostatic force on a unit of electric charge (due to a charge e located at distance L, of course). And as a basis for the electromagnetic system, he takes the relation :

$$\mathcal{H} = \frac{m}{L^2}, \tag{78}$$

where $\mathcal{H}$ is the "magnetic force" on a unit of magnetic pole (our "magnetic field").

In both cases, Maxwell simply substitutes in the dimensional versions of these relations the dimensions of the quantities concerned as a function of M, L, T and e on the one hand, and as a function of M, L, T and m on the other. He thus obtains the dimensions of *e* and *m* in each of the two systems:

Electrostatic system	**Electromagnetic system**
$\mathcal{F} = \frac{e}{L^2}$	$\mathcal{H} = \frac{m}{L^2}$
Size dimensions :	Size dimensions :
As a function of e: As a function of m :	As a function of e: As a function of m :
$[\mathcal{F}] = \frac{L\,M}{e\,T^2}$ $[\mathcal{F}] = \frac{m}{L\,T}$	$[\mathcal{H}] = \frac{e}{L\,T}$ $[\mathcal{H}] = \frac{L\,M}{m\,T^2}$
$[e] = e$ $[e] = \frac{L^2\,M}{m\,T}$	$[m] = \frac{L^2\,M}{e\,T}$ $[m] = m$
Therefore, since $[\mathcal{F}] = \frac{e}{L^2}$	Therefore, since $[\mathcal{H}] = \frac{m}{L^2}$
$\frac{L\,M}{e\,T^2} = \frac{e}{L^2}$; $\frac{m}{L\,T} = \frac{L^2\,M}{m\,T}\frac{1}{L^2}$	$\frac{e}{L\,T} = \frac{L^2\,M}{e\,T}\frac{1}{L^2}$; $\frac{L\,M}{m\,T^2} = \frac{m}{L^2}$
Hence :	Hence :

$[e]=L^{3/2}M^{1/2}T^{-1}$ and $[m]=L^{1/2}M^{1/2}$	$[e]=L^{1/2}M^{1/2}$ and $[m]=L^{3/2}M^{1/2}T^{-1}$

Table 3.2

Maxwell then gives a table showing the dimensions of the various quantities in each of the two systems.[80] It is through these that he defines v: the quotient of the electrostatic dimensions to the electromagnetic dimensions of the same quantity is a velocity (or its inverse) "whose absolute value is independent of the value of the fundamental units employed. This velocity is an important physical quantity, which we will denote by the symbol v."[81] In concluding his discussion, Maxwell thus gives considerations similar to those found in his article with Jenkin back in 1863, and which echo those discussed by Weber.

He strongly emphasizes the theoretical importance of v :

> This velocity...which expresses the relationship between electrostatic and electromagnetic phenomena, is a natural quantity with a definite value, and measuring this quantity is one of the most important areas of research in electricity.[82]

In his treatise, then, Maxwell attempts to develop more fundamental relations than before, correct independently of the system under consideration, from which the electrostatic and electromagnetic systems can be derived. He is also careful to note the

[80] Maxwell 1873: vol.2, chapter 10, 242. He also defines other quantities, quotients of those already discussed. Thus, he has the capacitance $\frac{e}{E}$, the self-induction coefficient $\frac{p}{C}$, the specific inductive capacitance of a dielectric $\frac{\mathcal{D}}{\mathcal{F}}$, the magnetic inductive capacitance $\frac{\mathcal{B}}{\mathcal{H}}$, the resistance $\frac{E}{C}$ and the resistivity $\frac{\mathcal{F}}{C}$, and indicates the dimensions of these quantities.

[81] "The absolute value of which does not depend on the magnitude of the fundamental units employed. This velocity is an important physical quantity, which we shall denote by the symbol v"; Maxwell 1873: vol.2, chapter 10, 243.

[82] "This velocity...which indicates the relation between electrostatic and electromagnetic phenomena is a natural quantity of definite magnitude, and the measurement of this quantity is one of the most important researches in electricity." Maxwell 1873: vol.2, 368.

structural similarities between the relations he obtains, even though he cannot draw any conclusions from them. Interestingly, some of these elements can be found in another contemporary publication. Indeed, we mentioned that several of the reports of the British Association Committee were published in 1873, by Jenkin. Maxwell and Jenkin's version of Appendix C of the 1863 report is published with a few modifications. Most are minor. Some concern notation, as already mentioned. Others concern dimensional results, which are much more developed in the 1873 version. As in the Treatise, and unlike the 1863 version, the electrostatic dimensions of magnetic quantities are tabulated, along with the ratio of electrostatic to electromagnetic units for each quantity.[83] Above all, we find some interesting relationships between quantities based on their dimensions, in particular products of quantities whose dimensions are those of an energy. The report states:

> The most instructive way of highlighting the relationships between these quantities is to arrange them in pairs, the product of each pair being a quantity of mechanical energy, work done per unit of time, energy per unit of volume, or work done in a unit of volume per unit of time.[84]

Finally, Maxwell and Jenkin give a table drawing a parallel between certain quantities having the same dimensions in different systems: thus, the electrostatic dimensions of the quantity of electricity are the same as the electromagnetic dimensions of the intensity of a pole. These quantities are grouped to form sets with the same dimension exponent for length and mass. [85]

[83] B.A. Report 1873: 90-91.
[84] "The most instructive method of exhibiting the relations of these quantities is to arrange them in pairs, the product of each pair being either a quantity of mechanical energy, or the work done in unit of time, or energy existing in unit of volume, or work done in unit of volume in unit of time." B.A. Report 1873: 91.
[85] B.A. Report 1873: 91.

Overall, it can be said that the contributions of the *Committee on electrical standards* in general and Maxwell in particular constitute a synthesis of those of Fourier, Gauss and Weber. From the former, this committee takes up what can be described as dimensional analysis: the concept of dimension in the context of the problem of changing units, and the fact that it is necessary to determine to what power the units of fundamental quantities intervene in the units of derived quantities. From the latter, they take over the notion of absolute measurements: they share their mechanistic view that it is possible to reduce the measurements of derived quantities to the fundamental mechanical quantities of length, time and mass. It's even conceivable that the way they expressed the dimensions of a quantity was inspired by Gauss's work (the latter resulting in an expression involving all three fundamental units at once). Finally, Maxwell and Jenkin borrowed from Weber the idea that several systems of units can be defined in electromagnetism, implying several dimensional formulas for the same physical quantity. Maxwell and Jenkin went beyond these contributions, developing original systems of units that took into account the influence of the environment.

3. 5 - The impact of Maxwell's contributions

The influence of Maxwell's Treatise was considerable, even if his theory had trouble gaining acceptance. In 1882, Rudolf Clausius (1822-1888) reported:

> The high esteem in which Maxwell is rightly held has not only led to his formulas being accepted as entirely correct in England, but has also led to them being adopted as they stand in the works of authors belonging to other nations. As far as the latter are concerned, I shall mention only the valuable *Leçons sur l'Electricité et le*

> *Magnétisme* (Paris, 1882) by Mascart and Joubert, and the useful work by Herwig *Physikalische Begriffe und absolute Maasse* (Leipzig, 1880).[86]

Maxwell's Treatise was the first source of dissemination of dimensional analysis in Great Britain in the 1870s, and in France and Germany in the 1880s. In addition to the research articles published in various journals, which we will examine in the following chapters, dimensional considerations made their appearance in physics treatises for students.[87] In the second half of the years1870, l Linnaeus Cunning's work in 1876, R. Wormell's in 1876 and Stanley Jevons' in 1877 are worthy of note. The fact that these treatises were conceived as introductions to physics or electricity testifies to the spread of dimensional analysis. At the same time, Joseph David Everett (1831-1904), secretary of the Committee for the Selection and Nomenclature of Dynamic and Electrical Units[88] (also appointed by the *British Association*) published a short treatise in 1875 (which he expanded in 1879), the aim of which was to set out the C.G.S. system, which contributed in part to this development (Jevons, in particular, refers to it). In France, treatises discussing dimensional analysis appeared in the early 1880s.[89]

Although beyond its scope, this growing importance of dimensional analysis remains largely motivated by the problem of electrical and magnetic units, and in particular by the existence of the two systems, electrostatics and electromagnetics. In Great Britain,

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[86] "The high estimation in which Maxwell is justly held could not fail to cause his formulæ not only to be accepted inEngland as correct throughout, but also to be adopted unaltered in the works of writers belonging to other nations. In regard to the latter, I will mention only Mascart and Joubert's valuable Leçons sur l'Electricité et le Magnétisme (Paris , 1882: and Herwig's useful book, *Physikalische Begriffe und absolute Maasse* (Leipzig , 1880)." Clausius 1882a: 382-383. Éleuthère Mascart and Jules Joubert.
[87] Atten 1992: 76.
[88] "Report of the Committee for the selection and nomenclature of dynamical and electrical units."
[89] Cumming 1876: 220-229; Wormell 1876, Wormell was principal of a London middle-class school; Jevons 1877: 325-328. Jevons is professor of political economy at University College London; Everett 1875; Everett 1879; Jevons 1877: 328; Mascart & Joubert 1882; Witz 1883; Wundt 1884.

we might mention the treatises by Robert M. Ferguson in 1882, Andrew Gray and John T. Sprague in Sprague in 1884, Alfred Daniell in 1885, Fleeming Jenkin in 1887, Andrew Gray again, as well as William Arnold Anthony and Cyrus F. Brackett in 1888 (who actually published in New York), Walter Larden (1855-?) in 1889; and in France, the famous treatise by Eleuthère Mascart (1837-1908) and Jules Joubert (1834-1910) in1882, l Aimé Witz's work in 1883, as well as the translation of a work by Wilhelm Maximilian Wundt (1832 - 1920) Wundt devoted to medical physics, in 1884.[90]

From the 1880s onwards, the question of systems of units came to the fore on the international scene. In 1881, the first of a series of international congresses dedicated to electricity was held in Paris - a dozen were to follow until 1911, but the first of these were the most significant in terms of the decisions taken. In 1881, William Thomson, Hermann Helmholtz, Rudolf Clausius, Gustav Kirchhoff and Wilhelm Weber gathered together. This first congress addressed the question of systems of units and standards. While the system selected by the Committee of the *British Association* was already followed by French telegraphists, it was not generally followed by physicists, even in France. In Germany, the electrostatic system is used, with millimeters and milligrams as the fundamental units. The situation was made all the more problematic by the fact that not only did the standards differ from one country to another, but above all their relationship was sometimes ill-defined, particularly in the case of resistance. The aim of the1881 a congress was to officially adopt an international system. After heated debate, the electromagnetic system was adopted.[91]

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[90] Ferguson 1882; Gray 1884; Sprague 1884; Daniell 1885 : 1885; Jenkin 1887; Gray 1888; Anthony & Brackett 1888; Larden 1889; Mascart & Joubert 1882; Witz 1883; Wundt 1884.
[91] Blondel 1990: 171-182.

This decision does not, however, relegate the electrostatic system to oblivion. As we have seen, treatises present both systems, for both theoretical and practical reasons. Hermann Helmholtz, the most eminent German physicist of the time, summed up the dilemma in 1882 in an article translated the same year in the pages of *Philosophical Magazine*.[92] Ideally, it would be preferable to have a single system, and Helmholtz preferred the electrostatic system:

> For purely scientific purposes, I would have preferred the electrostatic system hitherto employed, since this, I believe, best translates the essential analogies of phenomena into analogous formulae, and gives them the clearest and most intelligible expression. It is on this system...that most treatises on mathematical physics in this branch of science have hitherto been based.[93]

It was on the basis of such considerations that Clausius fervently defended the electrostatic system at the 1881 congress. Helmholtz himself used it in his memoirs. Nevertheless, duality seemed to him a necessary evil in the contemporary state of things, for reasons of experimental precision:

> For practical reasons, it was not possible to dispense with the use of these two systems of electrical measurement units, due to the fact that the determination of the factor that was used for the reduction of electrostatic units to electromagnetic units, namely Weber's critical speed, could not be carried out with the degree of precision that could be achieved in electromagnetic measurements on the one hand and electrostatic measurements on the other. For this reason, it was more advantageous to employ in each experimental research project the system of units in which the measured quantities could be most accurately processed.[94]

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[92] Helmholtz 1882a: 430-439; Helmholtz 1882b: 42-44.

[93] "Had the aim been purely scientific, I should have preferred the electrostatic system hitherto employed, since this, I think, best represents the essential analogies of the phenomena by analogous formulæ, and gives to them the clearest and most intelligible expression. It was on this system... that most of the physical mathematical treatises in this department of science have hitherto been based." Helmholtz 1882a: 431.

[94] "The employment of those two systems of electrical measures could not be dispensed with, for practical reasons, because the determination of the factor which had been used for the reduction of electrostatic to electromagnetic measures, namely Weber's critical velocity, could not yet be effected with the same degree of precision that could be attained within the sphere of electromagnetic measurements on the one hand and electrostatic measurements on the other. It was on this account more advantageous to employ in each experimental investigation that system of measures to which the quantities measured could be referred with the greater exactness." Helmholtz 1882a: 430-431.

Beyond its practical justification, this duality will give rise to conceptual controversies, as we'll see in the next chapter.

Chapter 4: Fundamental quantities

4. 1 - Introduction

As we have seen, in the wake of Maxwell's work, the spread of dimensional analysis began in the mid-1870s and was confirmed in the 1880s. A large number of publications dealt with the choice of fundamental quantities. It will be remembered that the British Association Committee in general, and Maxwell in particular, had adopted length, time and mass, following in the footsteps of Gauss and Weber. This was the generalized system, and is found in most treatises.[1] However, some treatises and many articles propose variations on this system.

There are several reasons for these proposals. Some are motivated by a concern for simplicity: the authors attempt to reduce the number of fundamental quantities, defining one of the three as a function of the other two. Others involve the use of fundamental quantities other than L, M and T.

4. 2 - Attempts to reduce the number of fundamental quantities

[1] In Great Britain: Cumming 1876; Jevons 1877; Tait 1884; Sprague 1884; Daniell 1885; Gray 1888; Larden 1889; and in France: Ganot & Maneuvrier 1894; Mascart & Joubert 1882; Witz 1883; Wundt 1884; Vaschy 1890.

Such an approach is far from new. In 1857, Weber proposed reducing the number of fundamental units to length alone:

> The units used in physics are of two kinds: *fundamental units* and *derived units*. In mechanics, where the forces are well known, units can be reduced to the three fundamental units of *space*, *time* and *mass*. In all areas of physics where the *law of gravitation* must appear, units can be reduced to the fundamental units of *space* and *time*; using the law of gravitation, we can derive the unit of *mass*. This is possible because we can take as a unit of mass that which, concentrated in one point, would exert a gravitational force on another mass located one unit away from the first, imparting to it, in one unit of time, a velocity equal to the quotient of one unit of length and one unit of time.
>
> It's interesting to note that this system of units can be simplified even further, and that it's possible to derive all the units of physics *from the single fundamental unit of length*, if we appeal to two fundamental natural laws, namely *the law of gravitation* and *the fundamental law for electrical effects*. Using the latter, it is possible to derive the unit of time from the unit of space. In fact, we can use, as a unit of time, the time it takes for two electric charges to move towards or away from a unit of length at a uniform relative speed, in such a way that they exert no action on each other. [2]

In fact, according to Weber's law,[3] there is a relative speed for which the force between two charges cancels out.

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[2] "Die in der Physik gebräuchlichen Maasse werden in *Grundmaasse* und *abgeleitete Maasse* eingetheilt. In der allgemeinen Mechanik, wo alle Kräfte einzeln als gegeben betrachtet werden, lassen sich alle Maasse auf die drei bekannten Grundmaasse für *Raum*, *Zeit* und *Masse* zurückführen. - In allen denjenigen Theilen der Physik, wo das *Gravitationsgesetz* vorausgesetzt werden darf, lassen sich alle Maasse blos auf die beiden Grundmaasse für *Raum* und *Zeit* zurückführen, aus denen mit Hülfe des Gravitationsgesetzes auch das Maass der *Masse* abgeleitet wird. Man kann nämlich diejenige Masse zum Maasse nehmen, welche, wenn sie in einem Punkte koncentrirt wäre, auf eine andere Masse in der Einheit der Entfernung nach dem Gravitationsgesetze eine Kraft ausübt, die ihr in der Zeiteinheit eine Geschwindigkeit ertheilt gleich der Längeneinheit in der Zeiteinheit.

Es ist nun interessant zu bemerken, dass auch dieses Maasssystem noch einer Vereinfachung fähig ist, und dass es möglich ist alle in der Physik gebrauchten Maasse *aus dem einzigen Grundmaass für Raum* abzuleiten, wenn man zwei Grundgesetze der Natur zu diesem Zwecke voraussetzen darf, nämlich ausser dem *Gravitationsgesetze ponderabler Massen* das *Grundgesetz der elektrischen Wirkung*. Denn mit Hülfe des letzteren kann auch *das Maass der Zeit aus dem Raummaasse abgeleitet werden*. Man kann nämlich diejenige Zeit zum Maasse nehmen, in welcher sich zwei mit gleichförmiger relativer Geschwindigkeit bewegte elektrische Massen um die Längeneinheit einander nähern oder von einander entfernen müssen, wenn sie nach diesem Gesetze gar keine Wirkung auf einander ausüben sollen." Weber & Kohlrausch 1857: in Werke: 667-668.

[3] According to Weber's law, the force between two electric particles (*e* and *e*') depends on their relative distance (*r*), but also on the first two derivatives of the latter with respect to time according to the formula :

$$f = \frac{ee'}{r^2}\left[1 - \frac{1}{C^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{2r}{C^2}\frac{\mathrm{d}^2 r}{\mathrm{d}t^2}\right]$$

Weber first uses Weber's law to relate T to L (with $c = 1$):

> If we choose the millimeter as the unit of length, assuming the fundamental law for electrical effects [i.e. Weber's law], this unit of length would lead to a unit of time equal to 1/439 450 millionths of a second, because when two electrical masses move uniformly towards or away from each other by one millimeter during this time, according to the fundamental law, they would have no effect on each other.[4]

Weber then uses the law of gravitation to derive a unit of mass based on the latter result. To do this, he expresses the gravitational acceleration at the Earth's surface in the new unit of time. He then imagines that the Earth's mass is concentrated in one point, and calculates the value of the gravitational acceleration that would take place at one unit of distance from this point:

> Once the unit of time has been derived from the unit of length in the way we have just described, we can go a step further and, assuming the law of gravitation, derive the unit of mass from these two units as well. According to the law of gravitation, the earth has a mass which, if concentrated in one point, would exert an acceleration of 9811 on another mass located at a distance equal to the earth's radius, if we take the millimeter as the unit of distance and the second as the unit of time. If we use as unit of time the new derived unit, which is 439450 million times smaller, then the unit of acceleration is $(439450)^2$trillion times larger, so that the value of the above acceleration in this new unit is :
>
> $$\frac{9811}{439450^2.10^{12}}.$$
>
> If we now take the Earth's radius to be 6370×10^6millimetres, then according to the law of gravitation, the Earth has a mass which, if concentrated in one point, would impart an acceleration of $\frac{9811.6370^2.10^{12}}{439450^2.10^{12}}$ to another mass located one unit away. From this we deduce that the unit of mass derived via the law of gravitation has the value of the fraction $439450^2/(9811.6370^2)$ (i.e. about half) of the Earth's mass if we take the millimetre as the unit of length and the unit already derived for time.

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where *C* is a constant with the dimension of a velocity. Darrigol 2000: 63.

[4] "Wählt man das *Millimeter* zum Raummaasse, so würde unter Voraussetzung des Grundgesetzes der elektrischen Wirkung aus diesem Raummaass ein Zeitmaass abgeleitet werden, welches der 439450 *Millionste Theil einer Sekunde* wäre; denn wenn zwei mit gleichförmiger relativer Geschwindigkeit bewegte elektrische Massen in diesem kleinen Zeitraume um 1 Millimeter sich einander nähern oder von einander entfernen, so üben sie nach dem Grundgesetz der elektrischen Wirkung gar keine Wirkung auf einander aus." Weber & Kohlrausch1857, in *Werke* : 668.

Finally, all other physical units can be derived from the millimeter and the two derived units of time and mass in the same way.[5]

An approach similar to Weber's can be found in a treatise published in 1879 by Joseph David Everett (1831-1904). This was the first work entirely devoted to physical units and constants from a dimensional perspective.[6] Everett, who had worked in William Thomson's laboratory in the 1860s (on the rigidity of solids), was at the time Professor of Natural Philosophy at Queen's College Belfast - as he was for most of his career. In his treatise, he aims to offer a consensual treatment of the subject. [7]

Everett explains that other choices are possible than length, time and mass (which we'll refer to as "quantities L, M, T"), but that this one happens to be the most convenient: "This particular choice is a matter of convenience rather than necessity."[8] Nevertheless,

---

[5] "Nachdem auf diese Weise aus dem Raummaass das Zeitmaass abgeleitet worden ist, kann ferner aus diesen beiden Maassen unter Voraussetzung des Gravitationsgesetzes auch das Maass der Masse abgeleitet werden. Es ist nämlich nach dem Gravitationsgesetze die Erde eine Masse, welche, wenn sie in einem Punkte koncentrirt wäre, einer anderen Masse in einer dem Erdhalbmesser gleichen Entfernung die Beschleunigung - 9811 ertheilt, wenn das Millimeter zum Raummaass und die Sekunde zum Zeitmaass gebraucht werden. Nimmt man nun statt der Sekunde das eben abgeleitete Zeitmaass, welches 439450 Millionen Mal kleiner ist, so ist das abgeleitete Beschleunigungsmaass 439 450^2Billionen Mal grösser, und es ist nach diesem grösseren Maasse obige Beschleunigung

$$\frac{9811}{439450^2.10^{12}}$$

Setzt man nun den Erdhalbmesser = 6370. $10^{(4)}$ (Millimeter), so ergiebt sich nach dem Gravitationsgesetze die Erde als eine Masse, welche, wenn sie in einem Punkte koncentrirt wäre, einer anderen Masse in der Einheit der Entfernung die Beschleunigung

$$\frac{9811.6370^2.10^{12}}{439450^2.10^{12}}$$

ertheilt, folglich ist eine Masse, welche $439450^2/\left(9811.6370^2\right)$ oder fast die Hälfte von der Erdenmasse beträgt, diejenige Masse, welche nach dem Gravitationsgesetze, unter Annahme des Millimeters als Raummaasses und mit Hülfe des daraus schon abgeleiteten Zeitmaasses, als *abgeleitetes Massenmaass* erhalten wird.
Aus dem Millimeter als Raummaass und aus dem daraus eben abgeleiteten Zeit- und Massenmaasse werden endlich alle übrigen in der Physik gebrauchten Maasse auf bekannte Weise abgeleitet".
Weber & Kohlrausch 1857: in Werke: 668-669.

[6] Everett 1879.

[7] Lees 2004: article about .Everett

[8] "This particular selection is a matter of convenience rather than of necessity." Everett 1879 : 15.

any choice of fundamental quantities must satisfy precise criteria. Most of these are of a practical nature:

> The most important considerations to guide us in the choice of fundamental units are the following:
> (1) They should be quantities that allow very precise comparison with other quantities of the same nature.
> (2) Such a comparison should be possible at any time. Standards must therefore be permanent - i.e., their size must not vary over time.
> (3) Such comparison should be possible at any place. Consequently, standards must not be such as to change size when transported from one place to another.
> (4) The comparison should be easy and straightforward.
> In addition to these experimental requirements, it is also desirable for the fundamental units to be chosen in such a way that the definition of the various derived units is easy, and their dimensions simple.[9]

In particular, Everett explains that force is not a satisfactory choice, as it does not satisfy the third condition. These criteria seem to be widely shared. Indeed, similar considerations can be found in the 1890 treatise by French telegraph engineer Aimé Vaschy.[10]

In addition, Everett requires that the fundamental quantities be irreducible to each other: "Any group of three *independent* units is theoretically sufficient."[11] Again, this is a generally accepted criterion. Vaschy explains:

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[9] "The following are the most important considerations which ought to guide the selection of fundamental units: -
(1) They should be quantities admitting of very accurate comparison with other quantities of the same kind.
(2) Such comparison should be possible at all times. Hence the standards must be permanent - that is, not liable to alter their magnitude with lapse of time.
(3) Such comparison should be possible at all places. Hence the standards must not be of such a nature as to change their magnitude when carried from place to place.
(4) The comparison should be easy and direct.
Besides these experimental requirements, it is also desirable that the fundamental units be so chosen that the definition of the various derived units shall be easy, and their dimensions simple." Everett 1879 : 16.

[10] The latter notes that the choice of L, M and T has the advantage that for mass, as for length, it is possible to define a universal standard. He even agrees with Everett on the disadvantages of force:
"To bring the scientific system of measurement into line with the commercial system, we could have made the unit of force one of the three fundamental units, and considered the unit of mass as a derivative. But as the commercial standard of force is represented by the weight of a certain well-defined mass (mass of a cubic decimeter of distilled water at $4^{(o)\,C}$) and as this weight, proportional to the intensity of gravity, has different values at different points on the globe, while the mass remains invariable, the desire to have a system acceptable to all countries dictated the choice of the system of fundamental units: L, M, T"; Vaschy 1890: 3.

[11] "Any three *independent* units are theoretically sufficient." Everett 1879: 15. Emphasis added.

> Any three units can be considered as fundamental, provided they are irreducible to each other. For example, we can adopt the M, V, F system, if this is advantageous from any point of view. But we could adopt neither of the two systems M, A, F and M, V, W, because knowledge of the units M of mass and A of acceleration entails knowledge of the unit F = MA of force, and since work is equivalent to a living force MV^2, the unit W of work follows from those of mass M and velocity V.[12]

Despite his remark, Everett believes that the number of fundamental quantities can be reduced by the law of universal gravitation. His reasoning is quite similar to Weber's, and we can assume that he was inspired by the latter, although he doesn't quote him. After deriving the dimensions $L^3M^{-1}T^{-2}$ from the gravitational constant, which he represents by C, he performs the following reasoning, which amounts to defining the units in such a way that C can be considered dimensionless:

> The equation $a = C\frac{m}{l^2}$ indicates that when $a = 1$ and $l = 1$, m must be equal to $\frac{1}{C}$; i.e., the mass that produces one unit of acceleration at a distance of one centimeter is 1.543×10^7grams. If this were taken as the unit of mass, with the centimeter and the second retained as the units of length and time, the acceleration produced by the attraction of any mass at any distance would simply be the quotient of the mass times the square of the distance.
> It is therefore theoretically possible to base a general system of units on just two fundamental units; one of the three fundamental units we have used up to now being eliminated via equation :
> $$\text{mass} = \text{acceleration} \times (\text{distance})^2,$$
> which gives the expression L^3T^{-2} for the dimensions of M.[13]

[12] Vaschy 1890: 3.

[13] "The equation $a = C\frac{m}{l^2}$ shows that when $a = 1$ and $l = 1$, m must equal $\frac{1}{C}$; that is to say, the mass which produces unit acceleration at the distance of1 centimetre is $1.543\text{x}10^7$grams. If this were taken as the unit of mass, the centimetre and second being retained as the units of length and time, the acceleration produced by the attraction of any mass at any distance would be simply the quotient of the mass by the square of the distance.
It is thus theoretically possible to base a general system of units upon two fundamental units alone; one of the three fundamental units which we have hitherto employed being eliminated by means of the equation
$$\text{mass} = \text{acceleration} \times (\text{distance})^2,$$
which gives for the dimensions of M the expression L^3T^{-2} ."
Everett 1879 : 60.

Two years after Everett's treatise, a paper by Frank Géraldy, an engineer from the Ponts et Chaussée engineering school in France, offered similar reflections.[14] He too begins by offering a justification for the choice of length, mass and time as fundamental quantities, but his arguments are more theoretical:

> Let's take...magnitudes by their origin:
> First of all, we find space, which requires a unit of length, which, as we know, is sufficient to measure surfaces and volumes.
> In this space is matter, and we need to know how much matter there is in a given space; this is called mass.
> Finally, we need to be able to note the order of phenomena and their duration, which is what we call time.[15]

Géraldy gives the impression of wanting to demonstrate that the choice of L, M, T as fundamental quantities corresponds to an absolute necessity arising from the very nature of the universe, in terms reminiscent of Newton's need to begin by defining absolute time and space.

However, Géraldy then seems to suggest that one of these quantities is not really fundamental. He too appeals to the law of gravitation, but it's the unit of time that he suggests deriving:

> So here we have three primary units: length, mass and time. Note, however, that the last of these depends on the other two, since we could take as a unit, for example, the time it takes for a mass to fall from a known height.[16]

Géraldy does not, however, go so far as to express T as a function of M and L.

[14] Géraldy 1881a: 89.
[15] Géraldy 1881b: 182.
[16] Géraldy 1881b: 182. Géraldy's method for deriving the unit of time may seem questionable: if we take as a unit the time taken for a mass to fall from a given height, for it to be universal, gravitational attraction would have to be universal too. However, if we're talking about the Earth's gravitational field, this is clearly not the case (apart from the fact that it is only valid forla Terre , it even varies from one part of its surface to another). Nevertheless, we can assume that Géraldy only means that the unit of time can be derived from the other two on the basis of the law of gravitation. The unit of time could then be defined as the time it takes for two unit masses, separated by a unit distance, to collide.

The following year, in 1882, the *Philosophical magazine* published a translation of an article by Helmholtz in which he derived all absolute units from natural constants - in this case, the gravitational constant, the density of water and the speed of light. Following Gauss's example, Helmholtz began by writing the law of gravitation in the form $F = \frac{m_1 m_2}{r^2}$, without a constant.[17] Combining this law with Newton's second law enabled him to establish a relationship between M, L and T, thus reducing the number of fundamental quantities to two:

$$\left[\frac{m^2}{r^2}\right] = [m.a], \tag{79}$$

hence :

$$\frac{\mathrm{M}^2}{\mathrm{L}^2} = \frac{\mathrm{M.L}}{\mathrm{T.T}}. \tag{80}$$

He thus obtains :

$$\frac{\mathrm{M}}{\mathrm{L}^3} = \frac{1}{\mathrm{T}^2}. \tag{81}$$

The way in which Helmholtz expresses this result suggests that he regards dimension formulas as synonymous with the physical nature of a quantity: "On the left is a density, on the right a function of time".[18] Such a perception guides the rest of his reasoning: having assimilated the term on the left to a density, he proposes to take the absolute density of pure water equal to1. L he unit of time is thus defined as a function of

[17] Carl Fredrich Gauss 1839.
[18] "On the left stands a density, on the right a function of time." Helmholtz 1882a: 432.

the latter. Having done this, only one arbitrary absolute measure remained: the unit of length. Helmholtz suggested defining it in terms of the speed of light. [19]

Two years later, in 1884, a publication with similarities to Helmholtz's article appeared in France. It was written by a certain Geza Szavardy, who published a series of articles in *La lumière électrique*, followed by a summary of his positions in 1887.[20] The beginning of his reasoning is quite similar to that just described, in that he uses Newton's second law and the law of gravitation to obtain $\mathrm{M} = \frac{\mathrm{L}^3}{\mathrm{T}^2}$. Szavardy then seeks to reduce the fundamental quantities to a single one. To do this, he uses two dimensional relations involving the quantity of electricity, $Q = F^{1/2}\mathrm{L}$, obtained using Coulomb's law with a purely numerical coefficient, and $Q = \frac{F^{1/2}\mathrm{L}}{V}$, obtained using Ampère's law with a purely numerical coefficient.[21] From these two relations, he deduces $V = 1$ and therefore L = T.

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[19] Helmholtz 1882a: 432.
More precisely, Helmholtz proposes to obtain the unit of time from the following - admittedly artificial - scenario: he considers a body in revolution near the surface of a sphere made of pure water [at 4° C], whose density D is assumed to be equal to1. L he absolute unit of time is then defined from the period of revolution of this body. As the relationship $\frac{\mathrm{M}}{\mathrm{L}^3} = \frac{1}{\mathrm{T}^2}$ indicates, this definition has the advantage of being independent of any consideration of length, since the period of revolution is independent of the sphere's radius:

$$T^2 = \frac{3\pi}{D}$$

Having thus defined the unit of time, we then obtain the unit of length from the following relationship (v representing the speed of light): . $v = \frac{\mathrm{L}}{\mathrm{T}} = \mathrm{L}\sqrt{\frac{D}{3\pi}}$

[20] Szavardy 1884: 321-326, 375-379, 413-417; Szavardy 1887: 401-408.

[21] His starting point was the following relationship: "Two quantities of static electricity of the same sign, subject to move in two parallel planes in such a way as to be always located at the feet of a perpendicular common to both planes, exert a mutual attraction proportional to the product of their numerical values and the square of their common velocity, and inversely proportional to the square of their distance."[21] , or $F \propto \frac{QQ'v^2}{l^2}$. The result is . $[Q] = \frac{[F]^{1/2}\mathrm{L}}{[V]}$

Implicitly, this means that he takes a system of units such that the speed of light is equal to one. With the previous result, $M = \frac{L^3}{T^2}$, Szavardy arrives at $M = L = T$, and concludes :

> This relationship tells us that only one of the three units of mass, length and time can be arbitrarily modified.... There is only one arbitrary unit. The dimensions of all other units are simply powers of this one.[22]

Szavardy introduces the notation "U" to designate this arbitrary unit and devotes the rest of his article to expressing the dimensions of the various physical quantities in terms of this single fundamental quantity.[23] Szavardy seems to have been carried away by a concern for unification and universality. At the end of his reasoning, he completely lost the notion that the dimensional expression of a quantity reflects its physical nature: two quantities of very different natures, for example mass and time, have the same dimensional exponent with respect to the universal unit U. Three years later, in his 1887 summary, Szavardy no longer mentions the possibility of reducing the fundamental quantities L, M and T to a single one.[24] In the meantime, he may well have abandoned the idea.

None of the attempts to reduce the number of fundamental quantities will win many converts. Some even questioned the validity of this approach. For example, Vaschy (to be discussed at length in a later chapter) asserted the irreducibility of L, M and T to each other:

> We have assumed that the three fundamental units L, M and T are irreducible to each other. They are indeed irreducible, at least in the current state of science, and attempts to link them by natural laws such as universal gravitation are based on illusions.[25]

[22] Szavardy 1884: 325.
[23] Szavardy 1884: 325-327.
[24] Szavardy 1887: 401-408.
[25] Vaschy 1890: 5.

Although Vaschy doesn't specify what he means, we can take a guess. Natural constants such as the speed of light or the gravitational constant certainly make it possible to define natural units of time and mass from the unit of length (for example). However, this possibility of natural choice of fundamental units does not abolish the freedom in principle to vary them independently of each other, since length, time and mass are fundamentally different quantities. Consequently, the units of the three fundamental quantities must always be distinguished, and their dimensions cannot be reduced to one another. Dimensional analysis exists only insofar as the difference in nature of the fundamental quantities is recognized and the distinction between their dimensions is preserved.

4. 3 - Fundamental quantities other than L, M and T

The proposals described so far involve reducing the number of fundamental quantities. But there are also proposals to use quantities other than length, mass and time, without reducing their number.

One of these suggestions is to use density rather than mass, i.e. L, D, T. It was first put forward in 1877 by W. Stanley Jevons, Professor of Political Economy at University College London. Jevons may have been inspired by the absence of mass in Fourier's original discussion, for at the start of his discussion he makes explicit reference to the latter and his *Analytical Theory of Heat.*[26] Jevons notes that, with density instead of mass, the dimensions of electrical quantities are simpler.[27] Two years later, in his 1879 treatise,

[26] W. Stanley Jevons, *The principles of science: A treatise on logic and scientific method*, London andNew York , 1877, 325.
[27] W. Stanley Jevons, *The principles of science: A treatise on logic and scientific method*, London andNew York , 1877, 327.

Everett made the same point. He tabulated the dimensions of the main electrical quantities as a function of L, D and T, in both the electrostatic and electromagnetic systems: [28]

	Electrostatics	Electromagnetic
Quantity	$D^{1/2} L^3 T^{-1}$	$D^{1/2} L^2$
Current	$D^{1/2} L^3 T^{-2}$	$D^{1/2} L^2 T^{-1}$
Capacity	L	$L^{-1} T^2$
Potential	$D^{1/2} L^2 T^{-1}$	$D^{1/2} L^3 T^{-2}$
Resistance	$L^{-1} T$	$L T^{-1}$

Table 4.1

Everett notes that the dimension exponents for L and T are all integers; this is undoubtedly where he finds the advertised simplicity.

In 1881, the Frenchman Gabriel Lippmann (1845-1921) gave the dimensions of electrical quantities in terms of force rather than mass. Lippmann's motivation stemmed from a controversy over the disadvantages of the dyne and its decimal multiples as units of force. Lippmann wanted to show that electrical quantities are not as easily affected by a change in force units as one might think. To this end, he tabulates their dimensions, taking this quantity as fundamental, in both systems of units:[29]

System	Current intensity	Resistance	Electromotive force	Capacity

[28] Everett 1879: 134.

[29] Lippmann 1881: 185. Lippmann obtains the formulas for dimensions in the electromagnetic system as follows (all symbols here represent dimensions of quantities). For current intensity, he starts from Coulomb's formula for magnetism and the force between a magnet and a current; the latter imply, respectively :

$$\frac{\mu^2}{L^2} = F \Leftrightarrow \mu = LF^{1/2} \qquad \text{and} \qquad \frac{\mu i}{L} = F \Leftrightarrow i = \frac{LF}{\mu} = \frac{LF}{LF^{1/2}} = F^{1/2}$$

For resistance, he posits that electrical work is equal to mechanical work:

$$ri^2 T = FL \qquad \Leftrightarrow \qquad r = \frac{FL}{i^2 T} = \frac{FL}{FT} = \frac{L}{T}$$

According to Ohm's law $e = r\,i$, the dimensions of the electromotive force are therefore the product of the two previous results.

As for electrical capacity, he takes the quotient of a charge by an e.m.f. and obtains the formula for charge from current, i.e. iT :

$$c = \frac{q}{e} = \frac{iT}{ri} = \frac{T}{r} = \frac{T}{L/T} = \frac{T^2}{L}$$

(Lippmann 1881: 183-185).

Electromagnetic	$i = \mathrm{F}^{1/2}$	$r = \frac{\mathrm{L}}{\mathrm{T}}$	$e = \frac{\mathrm{LF}^{1/2}}{\mathrm{T}}$	$c = \frac{I^2}{\mathrm{L}}$
Electrostatics	$I = \frac{\mathrm{LF}^{1/2}}{\mathrm{T}}$	$R = \frac{\mathrm{T}}{\mathrm{L}}$	$E = \mathrm{F}^{1/2}$	$C = \mathrm{L}$
Reports	$\frac{I}{i} = \frac{\mathrm{L}}{\mathrm{T}}$	$\frac{R}{r} = \frac{\mathrm{T}^2}{\mathrm{L}^2}$	$\frac{E}{e} = \frac{\mathrm{T}}{\mathrm{L}}$	$\frac{C}{c} = \frac{\mathrm{L}^2}{\mathrm{T}^2}$

Table 4.2

The resulting dimensions, as in Everett's L, D, T system, are characterized by integer exponents for L and T.

Another French scholar, Henri Pellat, also mentions the L, F, T system, albeit more briefly. For good measure, he also considers the L, M, F system and the F, M, T system:

> Instead of units of length, time and mass, we could take either units of length, time and force (the unit of mass would be a derived unit); or units of length, mass and force (the unit of time would be a derived unit); or units of time, mass and force (the unit of length would, in this case, be a derived unit).[30]

Although Pellat doesn't say so, it goes without saying that this is possible, since the formula for force dimensions in the L, M, T system involves these three quantities. Pellat does not dwell on these new systems, which he no doubt cites only to clarify the logic of the subject.

The systems L, D, T and L, F, T are not very different from L, M, T in that they also consist solely of mechanical quantities. There is, however, a suggestion to replace one of them by an electrical quantity. In1893, l 'ingénieur Claude Clavenad (1853-?) proposed using length, time and electric charge (L, T, Q). [31] He considered the magnetic pole to be a quantity derived from the latter.[32] Unfortunately, his reasoning and motives are unclear. He comes to the poor conclusion that $\mathrm{M} = \mathrm{L} = \mathrm{Q}$ - like Szavardy obtained . $\mathrm{M} = \mathrm{L} = \mathrm{T}$

[30] Pellat 1883: 46.
[31] Clavenad 1893: 552.
[32] Clavenad 1893: 551-555.

One of the most interesting works of the period is undoubtedly that of George Johnstone Stoney (1826-1911), vice-president of *the Royal Dublin Society*.[33] Stoney recalls the ideas put forward by Weber and Kohlrausch in 1857. But instead of reducing some of the fundamental units to others, he attempts to define new fundamental units using three "constants."

Stoney is motivated by the idea of making equations more elegant by eliminating superfluous coefficients wherever possible. He sees in such an exercise more than a matter of simplicity, but something profoundly necessary, a way of attaining a certain natural truth:

> When mathematicians apply the sciences of measurement to investigations of Nature, they find it convenient to choose the units of the various kinds of quantities they are dealing with in such a way as to eliminate any coefficients that can thus be avoided in their equations. With every advance in our knowledge of Nature, we see more clearly that it would be beneficial to our future progress to carry out such a simplification, not only with regard to certain classes of phenomena, but for the whole domain of Nature.[34]

Stoney notes that the question has hitherto been treated with a concern for the needs of mechanics, given the historical importance of this branch. If we stick to mechanics, it is possible to derive the units of the relevant physical quantities from those of any triplet of quantities:

> Until now, mathematical practice has been governed by the requirements of mechanics, a science in which (although not in its entirety) it is possible to derive the units of all other quantities from any three of them.[35]

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[33] Stoney 1881: 381-390.

[34] "When mathematicians apply the sciences of measurement to the investigation of Nature, they find it convenient to select such units of the several kinds of quantity with which they have to deal as will get rid of any coefficients in their equations which it is possible in this way to avoid. Every advance in our knowledge of Nature enables us to see more distinctly that it would contribute to our further progress if we could effect this simplification, not only with reference to certain classes of phenomena, but throughout the whole domain of Nature." Stoney 1881: 381.

[35] "Hitherto the practice of mathematicians has been governed by the demands of the science of mechanics, in the greater part (though not in the whole) of which science it is possible to derive the units of all the other kinds of quantity from any three which may be chosen." Stoney 1881: 382.

Stoney of course points out that the units normally used as fundamental are those of length, time and mass. Unlike Géraldy, he insists on the arbitrary nature of this choice:

> A system based on an arbitrary foundation is necessarily an artificial system.... In forming the artificial list of systematic units currently in existence, it has been customary to consider the units of length of time and mass as fundamental, and the others as derived; but *nothing prevents us from considering any three independent members of this list as fundamental, and deriving the others from them*.[36]

To remove this arbitrariness, he appeals to nature:

> The object of the present publication is to indicate that Nature presents us with three such [fundamental] units; and that if we take these as fundamental units..., we will bring our quantitative expressions into a more practical and undoubtedly more intimate relationship with nature as it is.[37]

The criterion on which Stoney bases his choice is a principle of universality:

> To do this, we need to select phenomena that occur in Nature as a whole, and are not associated with specific bodies.[38]

However reasonable this principle may seem, Stoney's three "fundamental units" are surprising for someone used to L, T and M. As for the first, Stoney's choice seems prescient:

> The first of Nature's absolute quantities to which I wish to draw attention is that remarkable speed of absolute value, independent of the units in which it is measured, which links all systematic electrostatic units with the electromagnetic units of the same series.[39]

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[36] "A system built in this way upon a foundation which is arbitrarily assumed is necessarily an artificial system." "In forming the existing artificial series of systematic units, it has been usual to regard the units of length, time and mass as fundamental, and the rest as derived; but *there is nothing to prevent our regarding any three independent members of the series as fundamental, and deriving the others from them*." Stoney 1881: 382 and 184. Italics by Stoney.

[37] "It is the aim of the present paper to point out that Nature presents us with three such units [i.e. alluded to in the previous quotation]; and that if we take these as our fundamental units instead of choosing them arbitrarily, we shall bring our quantitative expressions into a more convenient, and doubtless into a more intimate, relation with Nature as it actually exists." Stoney 1881: 384.

[38] "For such a purpose we must select phenomena that prevail throughout the whole of Nature, and are not specially associated with individual bodies." Stoney 1881: 384.

[39] "The first of Nature's quantities of absolute magnitude to which I will invite attention is that remarkable velocity of an absolute amount, independent of the units in which it is measured, which connects all systematic electrostatic units with the electromagnetic units of the same series." Stoney 1881: 384.

This is Weber's constant, which becomes the speed of light in Maxwell's theory. The reasons for Stoney's choice seem clear: this speed is a quantity independent of units, and as such is universal. This idea ties in with Weber's earlier idea of using the quotient C.

The second fundamental unit suggested by Stoney is a universal constant: the gravitational constant. This choice recalls Weber's proposal to use the law of gravitation to express the unit of mass as a function of those of time and length. Stoney writes:

> Here again, Nature presents us with a particular coefficient of gravitation, of an absolute magnitude independent of the units in which it is measured.[40]

The way Stoney expresses himself may seem awkward. Clearly, he doesn't mean that the numerical value of this constant remains the same whatever the chosen system of units, but simply that it is a universal constant in the usual sense of the term: its magnitude remains the same, whatever the physical circumstances considered. This is why it seems appropriate to use it as a standard.

Nevertheless, Stoney expects the idea of using G as a fundamental unit to meet with some resistance. We can even assume that this had already been the case, as he proposes in postscript to replace G with a unit of mass:

> Many people find it difficult to conceive of G_1 as a unit. It's possible to avoid G_1 and substitute $M_{(1)}$, with M_1 defined as the mass that attracts an equal mass at a distance with the same force as that which two units of electricity..., separated by the same distance, would exert on each other.[41]

[40] "Again, Nature presents us with one particular coefficient of gravitation, of an absolute amount independent of the units in which it is measured." Stoney 1881: 384.

[41] "Postscript.-Many persons find it difficult to conceive of G_1 as a unit. G_1 may be avoided and M_1 substituted for it, if M_1 be defined as a mass such that it attracts an equal mass at a distance with the same force with which two units of electricity... would, if placed at the same distance asunder, act on each other." Stoney 1881: 390.

The resistance to which Stoney alludes here is probably due to the fact that it's hard to see how G could be used as a concrete standard: there is no physical quantity to which numerical values can be assigned by comparing it with G, taken as a concrete unit.

Stoney's third fundamental unit is a quantity of electricity. On this point, he departs from the program of reduction to purely mechanical quantities initiated by Gauss and Weber. His problem was to find such a quantity that was invariant and natural. Unlike the scientists discussed so far, Stoney does not resort to Coulomb's electrostatic law. He uses an entirely different phenomenon: electrolysis. The reason for this was that, at the time, proponents of atomism believed that the quantity of electricity exchanged when a chemical bond is broken is a universal constant independent of the bodies considered. Stoney offers the following definition:

> La Nature presents us, in the phenomenon of electrolysis, with a single, well-defined quantity of electricity, independent of the bodies in question. In order to clarify this, I will express Faraday's law in the following way, which, as I will show, gives it greater precision: - *For each chemical bond that is broken in an electrolyte, a certain quantity of electricity passes through the electrolyte, which is the same in all cases*. I'll call this particular quantity of electricity $E_{(1).}$ If we make it our unit of electricity, we'll probably have taken an important step forward in our study of molecular phenomena.[42]

Stoney presents a lengthy discussion to demonstrate his version of Faraday's law, and to determine the value of E_1. We won't report it here, as it doesn't involve dimensional analysis. [43]

[42] "Nature presents us, in the phenomenon of electrolysis, with a single definite quantity of electricity which is independent of the particular bodies acted on. To make this clear I shall express "Faraday's law "in the following terms, which, as I shall show, will give it precision, viz.: - *For each chemical bond which is ruptured within an electrolyte a certain quantity of electricity traverses the electrolyte, which is the same in all cases*. This definite quantity of electricity I call E_1. If we make this our unit quantity of electricity, we shall probably have made a very important step in our study of molecular phenomena." Stoney 1881: 385. Italics by Stoney.

[43] Stoney 1881: 385-388.

To sum up, Stoney accepts three fundamental units: the speed of light V_1, the gravitational constant G_1 and the quantity of electricity E_1 exchanged when a chemical bond breaks (also considered a universal constant, independent of the bodies considered). He insists on the natural character of these quantities, and truly seems to regard them as a choice that is self-evident - almost that of nature itself:

> We have very good reason to suppose that in V_1, G_1 and E_1 we have three units of a systematic series of units that are in a profound sense the units of nature, and are intimately linked to the work taking place in its mighty laboratory.[44]

Stoney then set about expressing the value of the units of length, time and mass, now derived units, in the metric system.[45] To do this, he turned to dimensional analysis.

Stoney begins by giving the value of V_1, G_1 and E_1 in the usual units:

$V_1 = 3.10^8$ meters per second
$G_1 = 2/3.10^{(-13)\ \mu(_1)}$ where μ_1 corresponds to 1 m^3/s^2kg
$E_1 = 10^{(-22)\ e(_1)}$ where e_1 corresponds to 100 amperes

For convenience, the above numerical values are referred to as A, B and C respectively:

$V_1 = A$ m/s	$A = 3.10^8$
$G_1 = B\ \mu_1$	$B = 2/3.10^{-13}$
$E_1 = C\ e_1$	$C = 10^{(-22)}$

It expresses the dimensions of its three new fundamental quantities in terms of L, T, M :

speed V_1: $$\frac{L}{T} \tag{82}$$

attraction "coefficient" G_1: $$\frac{L^3}{MT^2} \tag{83}$$

electromagnetic quantity of electricity E_1: $$\sqrt{LM} \tag{84}$$

[44] "We have very good reason to suppose that in V_1, G_1 and E_1 we have three of a series of systematic units that in an eminent sense are the units of nature, and stand in an intimate relation with the work which goes on in her mighty laboratory." Stoney 1881: 385.
[45] Stoney 1881: 389-390.

It deduces the following relationships:

$$\frac{L_1}{T_1} = A\frac{l_1}{t_1},,,\frac{L_1^{\ 3}}{M_1T_1^{\ 2}} = B\frac{l_1^{\ 3}}{m_1t_1^{\ 2}}\ \sqrt{L_1M_1} = C\sqrt{l_1m_1} \qquad \textbf{(85)}$$

where L_1, $T_1$and M_1represent the derived units of length, time and mass of the system he advocates, and l_1, $t_1$and $m_1$the units of length, time and mass in the metric system (meter, second and gram). Stoney then solves these equations for L_1, $T_1$and $M_1$and obtains :

$$L_1 = \frac{C\sqrt{B}}{A}l_1,,.T_1 = \frac{C\sqrt{B}}{A^2}t_1\ M_1 = \frac{CA}{\sqrt{B}}m_1 \qquad \textbf{(86)}$$

He then translates these equations into numerical terms, obtaining :

$L_1= 10^{-37}$meters
$T_1= 1/3.10^{-45}$seconds
$M_1= 10^{(-7)}$ grams

Stoney concludes his work with some characteristic remarks:

> This seems to be the best attempt we can now make to determine these remarkable units. In the series to which they belong, all electrostatic units will be identical to the corresponding electromagnetic units; all the forces of Nature that we know obey the inverse square law, whether they result from gravitation, electricity or magnetism, will be expressed without coefficients; and the chemical bond, which seems to be the unit of concrete Nature, will then find itself in its true relation to physics.[46]

So he's happy to have achieved his goal.

Some fifteen years later, in 1897, in *L'Eclairage Electrique*, a certain Ricardo Malagoli suggested using the quantity of electricity Q, the quantity of magnetism m and the work W as fundamental quantities.[47] He, too, thus moved away from the mechanistic credo. Nevertheless, his ideas are in line with others of Germanic origin: Malagoli shares

[46] "This appears to be the best attempt we can yet make to determine these remarkable units. In the series to which they belong all the electrostatic units will be identical with the corresponding electromagnetic units, all the forces of Nature that are known to obey the law of the inverse square, whether they arise from gravitation, electricity, or magnetism, will be expressed without coefficients, and the chemical bond, which seems to be the unit of concrete Nature, is brought into its proper relation to physics." Stoney 1881: 390.
[47] Malagoli actually uses the symbols Q_e, $Q_m$and W, respectively. Malagoli 1897: 535-539.

the notion of taking energy as the fundamental unit with the German champion of energeticism Wilhelm Ostwald (1853-1932).[48] But his motivations differ from those of Ostwald, who "proposes to replace the unit of mass by the unit of energy for the reason that energy has a more secure existence than that of matter."[49] Of the three fundamental quantities in use, Malagoli has the least confidence in time. Discussing the reasons for using length, mass and time, he states:

> From an experimental point of view, we can invoke the ease with which we can establish and reproduce a standard of these two units. For the third, easy reproduction is impossible. The third, on the other hand, will always be a unit *deduced* from one of the two movements of the earth, preferably from that of an ideal planet moving uniformly on the ecliptic plane and whose center would coincide only with that of the earth for the two equinoxes.[50]

These remarks are reminiscent of Everett's suggestions in 1879 and Géraldy's in 1881 to make time a derived quantity. [51]

Malagoli first suggested replacing the unit of time by force or work, since these two quantities are predominant in most physical phenomena.[52] Finally, he proposed Q, m and W as the fundamental quantities, although he was aware of the arbitrariness of this choice; he recognized that it was sufficient for the three fundamental quantities to be capable of generating all the others. In the L, M, T electromagnetic system, the quantities Q, m and W have the dimensions

$$Q : L^{(1/2)\ M(1/2)}$$
$$m : L^{(3/2)\ M(1/2)\ T(-1)}$$
$$W : L^2 M\ T^{-2}$$

[48] Ostwald 1892. However, Malagoli says he had the idea independently. (Malagoli 1897: 536).
[49] Malagoli 1897: 536.
[50] Malagoli 1897: 536.
[51] Everett 1879: 60; Géraldy 1881b: 182.
[52] Malagoli 1897: 536.

Since Q, m and W involve the three quantities L, M and T, physical quantities that can be expressed in the L, M, T system can also be expressed in the Q, m, W system. Malagoli tabulates the dimensions of various physical quantities in this new system.[53]

He claims to have tried other systems, but prefers Q, m, W because "no other [system] offers simpler dimensions, for derived units, than this."[54] Like Everett in 1879, he considers equations with integer exponents to be simpler. Had he been able to find a system that gives rise to equations of dimensions whose exponents are both integer and positive, he would have preferred the latter; but he asserts that no such system exists.[55]

Malagoli also uses a less rudimentary simplicity criterion:

> A greater number of these quantities [i.e. the various physical quantities] are expressed in terms of just two of the three fundamental units, which means that most physical phenomena have closer relationships with the new units than with those of length, mass and time.[56]

Indeed, of the twenty-eight quantities he considers, eleven are expressed as functions of two fundamental quantities in the L, M, T system, and fifteen in the Q, m and W system - (and only thirteen relate exclusively to electrical or magnetic phenomena). Taking into account the number of fundamental quantities appearing in the dimensional formula of a physical quantity seems an interesting notion as a criterion of simplicity. However, the idea that it reflects "closer relations" between a physical phenomenon and the said fundamental quantities is questionable.

Malagoli's proposed system doesn't seem to have met with much success. He himself acknowledges its impracticality:

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[53] Malagoli 1897: 538.
[54] Malagoli 1897: 538.
[55] Malagoli 1897: 538. Malagoli's idea is somewhat surprising: it amounts to saying that the system L, M, T is unsatisfactory because the velocity there has dimension LT^{-1}.
[56] Malagoli 1897: 538.

> The adoption of this new system of fundamental units would be...to be recommended, if the realization of the standards of these new units were as simple as that of the old ones.[57]

What's more, as we'll see in the next chapter, it's easily open to theoretical criticism.

In the last two decades of the 19th century, despite the frequent use of length, mass and time as fundamental quantities, there were several alternative proposals. Some attempted to reduce the number of fundamental quantities, attracted by the greater simplicity such a system would represent. Others suggest replacing some of these fundamental quantities with others, or at least noting that some alternatives offer advantages that L, M, T don't, such as a certain simplicity in the dimensional formulas for electrical quantities. Other proposals are linked to energetic conceptions. There's even the suggestion not to reduce all physical quantities to those of mechanics, but to introduce charge as a fundamental quantity, breaking with the Weberian tradition. The hegemony of the L, M, T system, although confirmed in practice, is not a barrier to the investigation of alternatives. These discussions show that, in a context where dimensional analysis is entering the mainstream, the study of its foundations is alive and well.

[57] Malagoli 1897: 538.

Chapter 5: The electrostatic dimensions of the magnetic pole - a controversy

5. 1 - Introduction

The first controversy concerning dimensional issues took place at1882 a . It took place mainly in Great Britain, in the pages of *Philosophical magazine*, and concerned the dimensions of the magnetic pole.[1] This was an important issue for two reasons. Firstly, whereas up until then the same electromagnetic quantity only took on different dimensions in different systems of units, in this new case we find two different dimensional formulas for the same quantity *in the same system of units* (the electrostatic system), which many scientists considered problematic. Secondly, the issue goes far beyond the magnetic pole: since the dimensions of the magnetic pole are used to determine the dimensions of all other magnetic quantities, the entire magnetic part of the electrostatic system is at stake.

5. 2 - Origin of the controversy: An article by Clausius

[1] Although the controversy originated in a German article, and it is possible that a similar controversy took place in the Germanic world (either in parallel, or following the British reaction), the present study is limited to English and French sources.

The controversy originated in a publication by Rudolf Clausius, published in German in March 1882 and soon translated into the *Philosophical magazine.*[2] Clausius came up with a dimensional formula for the magnetic pole in the electrostatic system that differed from Maxwell's formula.

Clausius' article is intended as a kind of summary on the question of the two systems, electrostatics and electromagnetics. Clausius takes Maxwell's Treatise, which he praises, as his main reference on the subject. There were two main reasons for his own work: firstly, he wished to present a more coherent account than Maxwell's, in that the latter dealt with the question of systems of units in different parts of his work. Secondly - and this is where the controversy arises - Clausius disagrees with Maxwell's use of a certain equation.

Clausius begins by explaining the difference between electrostatic and electrodynamic systems. He stresses that in one case, the forces involved do not depend on the motion of the entities concerned, whereas in the other they do:

> To measure electricity, we need to use the forces it exerts. There are essentially two kinds of force: - firstly, forces independent of its motion, which quantities of electricity always exert on each other whether at rest or in motion; secondly, forces that result solely from its motion. The former are called *electrostatic* forces, the latter *electromagnetic*. Electromagnetic forces include magnetic forces, if we follow Ampère and derive the explanation of magnetism from small electric currents inside the magnet. Now, of these two forces, one or the other can be used to measure electricity; hence the two systems of units of measurement, the first of which is called electrostatics, while the second...is generally called electromagnetics, although it would be more rational to call it electrodynamics.[3]

[2] Clausius 1882a; the translation of this article appeared in the *Philosophical* Magazine in June: Clausius 1882b: 381.

[3] "For the measurement of electricity we must employ the forces exerted by it. These are of two essentially different kinds: -first, the forces independent of its motion, which amounts of electricity always exert upon one another whether they are at rest or in motion; secondly, the forces arising only through the motion. The former are named the *electrostatic*, and the latter the *electrodynamic* forces. To the electrodynamic we must reckon the magnetic forces, if we with Ampère derive the explanation of magnetism from small electric

Clausius designates the units in the electrostatic system by the subscript "s" for static, and those in the electromagnetic system by the subscript "d" for dynamic. He begins by deriving the formulas for the dimensions of the most important quantities in each of the two systems, in the sense that their units serve as the "basis for all the others,"[4] i.e. electric charge in the electrostatic system, and the quantity of magnetism (the intensity of a pole) in the electromagnetic system. Starting from the appropriate Coulomb formula in each case ($f = \frac{e_1 e_2}{r^2}$ and $f = \frac{m_1 m_2}{r^2}$ respectively), Clausius obtains :[5]

$$[e_s] = \mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-1}, \qquad [m_d] = \mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-1}. \tag{87}$$

He then set about determining the dimension formulas for these same quantities, but in the other system: he wanted to obtain $[e_d]$ and $[m_{(s)}]$. It is this discussion that lies at the root of the disagreement.

Clausius' reasoning is simple: since we know $[e_s]$ and $[m_{(d)}]$, all we need to do is find a relationship between e and m that is correct in both systems. He based this on Ampère's hypothesis that an electric current in a circuit is equivalent to two magnetic sheets of opposite magnetism (one north, the other south). Clausius is quick to point out that Maxwell agrees with this equivalence:

> [I adopt] Ampère's well-known proposal concerning the substitution of a galvanic current by two magnetic surfaces, which was also adopted by Maxwell in its

currents taking place in the interior of the magnet. Now, of these two forces, we can apply either the one or the other to the measurement of electricity; and thence arise those two systems of measures, of which the former is called the electrostatic, while the latter... is ordinarily named the electromagnetic, but would be more rationally called the electrodynamic." Clausius 1882b: 382-383.

[4] "[serve] as the basis of all the others." Clausius 1882a: 383.

[5] Clausius 1882b: 383-384.

greatest generality and without reference to any particular system of units of measurement.[6]

Clausius therefore considers a plane circuit, delimiting a surface of unit area, and imagines a plane figure of the same shape placed on a plane parallel to that of the circuit, and located at an infinitesimal distance ε from the plane of the latter. He imagines both figures covered with an equal amount of magnetism, but North in one case and South in the other:

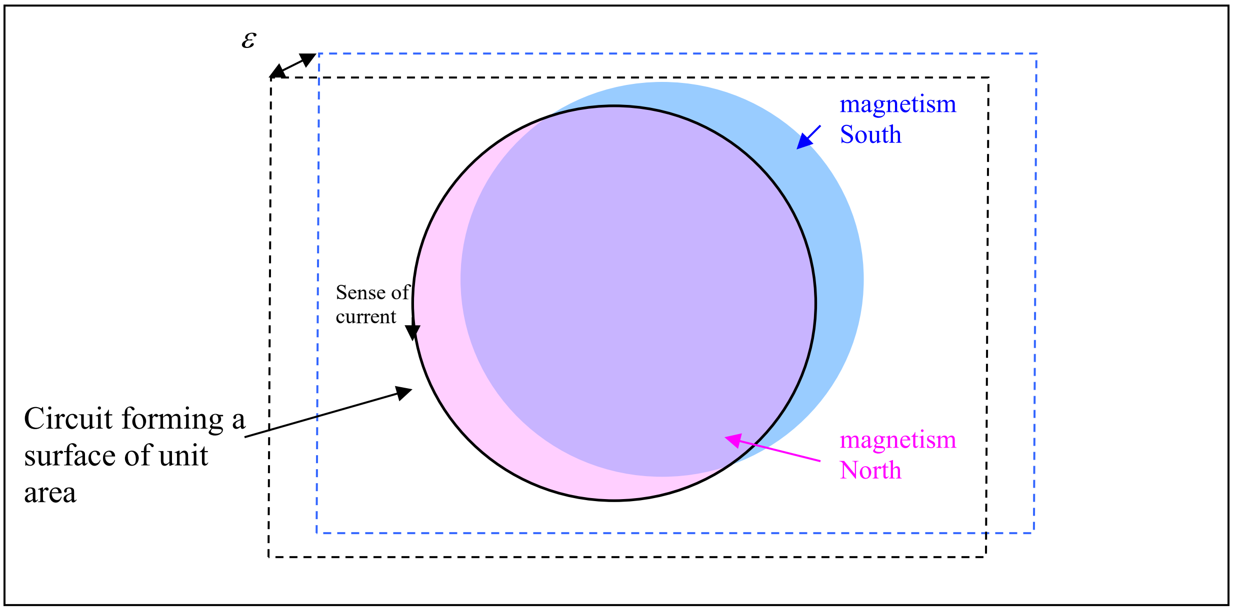

Figure 5.1

For the sheets to produce magnetic forces similar to those produced by the circuit, the quantities must satisfy the following relationship:

current intensity in the circuit $\times$ area of the surface bounded by the circuit

= amount of magnetism of one of the laminations $\times$ distance between the two surfaces

[6] "The well-known proposition of Ampère respecting the substitution for a closed galvanic current of two magnetic surfaces, which has also been adopted by Maxwell in its utmost generality and irrespective of any particular system of measurement." Clausius 1882b: 384. Maxwell 1873: vol.2, part 3, chapter 3.

For dimensions, this means $[e]\mathrm{T}^{-1}\times\mathrm{L}^2=[m]\times\mathrm{L}$ or $\frac{[m]}{[e]}=\mathrm{LT}^{-1}$.

Clausius declares:

> This equation, which simply expresses Ampère's relationship between magnetism and electric currents, must be used in any system of measurement units.[7]

Clausius therefore uses it to determine [e_d] and [$m_{(s)}$], thus obtaining, by substituting the results obtained previously for [e_s] and [m_d] :

$$[e_d]=\mathrm{M}^{1/2}\mathrm{L}^{1/2},\qquad [m_S]=\mathrm{M}^{1/2}\mathrm{L}^{5/2}\mathrm{T}^{-2}. \tag{88}$$

However, Maxwell does not obtain the same result for [m_s]: he obtains . $\mathrm{M}^{1/2}\mathrm{L}^{1/2}$ [8]

Clausius notes that Maxwell uses the following two relations:[9]

$$[pC]=\frac{\mathrm{L}^2\mathrm{M}}{\mathrm{T}^2}\quad\text{and}\qquad \left[\frac{e}{C}\right]=\mathrm{T} \tag{89}$$

where C represents the current intensity. Clausius then identifies [p] with [m], not without pointing out that Maxwell does the same.[10] After this substitution, multiplying these two equations member by member gives $[e\,m]=\mathrm{ML}^2\mathrm{T}^{-1}$ instead of $\frac{[m]}{[e]}=\mathrm{LT}^{-1}$. Clausius considers this reasoning to be incorrect, as he considers the $[e\,m]=\mathrm{ML}^2\mathrm{T}^{-1}$ relationship to be inadmissible in the static case:

> The way Maxwell arrives at this equation lies in the fact that he brings into the calculations the force exerted by the current on a magnetic pole, in the same way that, to obtain [$[e_S]=\mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-1}$ and $[m_d]=\mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-1}$], we brought in the force acting between two units of electricity, and that acting between two units of magnetism, respectively. However, the force that a current exerts on a magnetic pole is electrodynamic; and it follows that an equation whose derivation is based

[7] "This equation, which is only an expression of the relation established by Ampère between magnetism and electric currents, must hold good for every system of measurement" Clausius 1882b: 384-385.
[8] See chapter 3.
[9] See Chapter 3, equations (36) and (38)).
[10] Indeed, this seems implicitly to be the case. Maxwell 1873: vol.2, chapter 10, 241, § 624.

on this force can only be considered valid in the dynamic system based on electrodynamic forces, and not in the static system based on electrostatic forces.[11]

The problem is not limited to the dimensions of the quantity of magnetism: since the dimensions of the other magnetic quantities are derived from them, the electrostatic system as a whole is affected:

> As a result of Maxwell's incorrect formula for the static unit of magnetism, the formulas for other units in the static system of units of measurement, which depend on the unit of magnetism, have also been rendered incorrect.[12]

5. 3 - Everett starts the debate

The controversy continued in England the following month with two articles by Everett.[13] Everett begins with a brief summary of Clausius' reasoning, from which he derives the following relationships (which he specifies are to be understood in a dimensional sense):

$$\text{Pole} \times \text{Length} = \text{Current} \times (\text{Length})^2, \qquad \textbf{(90)}$$

hence :

$$\text{Pole} = \text{Current} \times \text{Length}. \qquad \textbf{(91)}$$

He points out that this is true for Clausius whatever the system of units considered. However, sticking to the electrostatic system, he gives the dimensions of the current in the latter as $\mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-2}$ and deduces the dimensions of the magnetic pole:

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[11] "The way in which Maxwell arrives at his equation rests upon his bringing into calculation the force exerted by a current upon a magnetic pole, in a similar manner to that in which we, in deducing [$[e_S] = \mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-1}$ and $[m_d] = \mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-1}$], have brought in the force acting between two units of electricity, and between two units of magnetism, respectively. The force, however, which a current exerts upon a magnetic pole is electrodynamic; and from this it follows that an equation of which the deduction is based upon this force can be regarded as valid only in the dynamic system founded upon the electrodynamic forces, and not in the static system based on the electrostatic forces." Clausius 1882b: 386.

[12] "By Maxwell's incorrect formula for the static magnetism-unit the formulæ of other units in the static system of measures, dependent on the unit of magnetism, have been rendered also incorrect." Clausius 1882b: 387.

[13] Everett 1882: 376-377 and 431-434.

$$M^{1/2}L^{3/2}T^{-2} \times L = M^{1/2}L^{5/2}T^{-2}$$

(92)

Everett considers Clausius' reasoning to be "beyond reproach" and gives a further argument in its favor. In the electromagnetic system, a pole has the dimension $M^{1/2}L^{3/2}T^{-1}$ and the current has the dimension $M^{1/2}L^{1/2}T^{-1}$. We therefore have [pole] = [current]× L. The fact that the result used by Clausius in the electrostatic system can be found in the electromagnetic system seems to Everett to corroborate the latter[14] - which may seem strange given that this relationship is, as we have just seen, a direct consequence of the equivalence of a circuit to a magnet, and therefore independent of the system considered, as Everett himself points out.

Everett's publication may not have been original, but it did set off a debate in Great Britain: not only did Everett draw attention to Clausius's ideas, he explicitly encouraged his readers to intervene. In particular, he declares his inability to understand how Maxwell derives his result, and invites his audience to propose an explanation.[15] This difficulty undoubtedly stems from a specific stage in Maxwell's reasoning. As we have seen,[16] Maxwell first gives himself a number of dimensional relations, which are either products of quantities whose dimensions are those of energy or energy density, or quotients of quantities whose dimensions are particularly simple - T, L and L^2. From these relations, he derives the dimensions of electrical and magnetic quantities as a function of both L, M, T and e, the quantity of electricity, and L, M, T and m, the quantity of magnetism. When he derives the dimensions of a magnetic pole in the electrostatic system, it's from two of these dimensional formulas and the equation

[14]Everett 1882 : 376-377.
[15]Everett 1882 : 376.
[16] See chapter 3.

$$\Phi = \frac{e}{L^2} \quad \textbf{(93)}$$

whereΦ , the "electromotive force at a point," corresponds to the electrostatic force on a unit of electric charge (see Chapter 3). This part of Maxwell's reasoning is very clear. Less clear, however, is the exact way in which he derives the aforementioned dimensional formulas from the relations he first gives himself. What's more, Everett probably wants a demonstration that allows us to obtain the dimensions of a pole more directly, without going through Maxwell's dimensional relations and his formulas with four fundamental quantities.

Everett's appeal is heard. In May, he published three different demonstrations that he had received.[17] The first two are derivations of Maxwell's result, while the third recovers Clausius' result. These various reactions led Everett to draw conclusions about the nature of the divergences:

> The definitions of the unit of quantity of electricity, the unit of current, the unit of electromotive force (or potential difference), and the unit of resistance (which can all be described as purely electrostatic definitions), pose no problem, and are accepted by all. Disagreement begins when we try to express magnetic quantities in the electrostatic system; and different results are obtained depending on the relationship between magnetism and electricity we choose as the guiding principle in our definitions. Strictly speaking, there is no such thing as an electrostatic unit of a magnetic quantity, since magnetism and its relationship to electricity lie outside the realm of electrostatics.[18]

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[17]Everett 1882 : 431-434.

[18] "The definitions of the unit quantity of electricity, the unit current, the unit electromotive force (or difference of potential), and the unit resistance (all of which may be called purely electrostatic definitions), are not in question, but are accepted by all parties. The divergence begins when we attempt to express magnetic quantities in an electrostatic system; and different results may be obtained according to the particular relation between magnetism and electricity which we select as the guiding principle in our definitions. In a strict sense there is no such thing as an electrostatic unit of any magnetic quantity; since magnetism and its relations to electricity lie outside the domain of electrostatics." Everett 1882 : 431.

The first demonstration was by Irish physicist Joseph Larmor (1857-1942). It's based on what Everett calls the "galvanometer law".[19] This gives the force exerted on the pole of a magnetized needle by a current flowing through the coil of a galvanometer:

$$\text{force} \ = k_1 \frac{\text{longueur du fil} \times \text{intensité du courant} \times \text{pôle}}{(\text{distance entre le fil et le pôle})^2} \tag{94}$$

Everett introduces the constant k_1, absent in Larmor's work, for a reason that will become clear later. Larmor thus obtains :

$$\text{pole} = \frac{\text{force} \times (\text{distance entre le fil et le pôle})^2}{\text{longueur du fil} \times \text{intensité du courant}} \tag{95}$$

The dimensions of the current intensity in the electrostatic system being, according to general opinion, $M^{(1/2)\,L(3/2)\,T(-2)}$, we obtain for the dimensions of a pole :

$$[\text{pole}] = .\frac{MLT^{-2} \times L^2}{L \times M^{1/2} L^{3/2} T^{-2}} = M^{1/2} \frac{L^3}{L^{5/2}} = M^{1/2} L^{1/2} \tag{96}$$

Clausius' criticism that Maxwell based the electrostatic system on an electrodynamic force applies to this interpretation of Larmor. Everett's reaction is interesting: while this is certainly the case, he says, all other definitions of the magnetic pole unit in the electrostatic system, including that of Clausius himself, have the same flaw.[20] This last remark is only partially justified: it's true that Clausius takes a circuit into consideration in his reasoning, but the equation he derives from it expresses *the equivalence* between this dynamic situation and the magnetic situation (which he himself considers to be electrodynamic); Larmor's relation, on the other hand, expresses *the influence* of a current on a magnet.

[19]Everett 1882 : 431.
[20]Everett 1882 : 432.

The second derivation presented by Everett was sent to him by another of Maxwell's Irish disciples, George Francis FitzGerald (1851-1901), Professor of Natural Philosophy at Trinity College Dublin.[21] It is based on the "magnetoelectric law," according to which the electromotive force produced when a conductor is moved in a magnetic field is proportional to the conductor's length, speed and field strength:

$$\text{f.e.m.} \quad \propto \quad L \times v \times B. \tag{97}$$

FitzGerald then used the idea that the intensity of the field due to a pole is proportional to the intensity of the pole and inversely proportional to the square of the distance :

$$\text{f.e.m.} \quad \propto \quad L \times v \times \text{ pole / distance}^2 \tag{98}$$

or, in Everett's version,

$$[\text{f.e.m.}] = [k_2] \times \mathrm{L} \times \mathrm{LT}^{-1} \times \text{ [pole] }, \times \mathrm{L}^{-2} \tag{99}$$

with $k_2 = 1$. Hence

$$[\text{pole}] \; = \frac{[\text{f.e.m.}]}{\mathrm{L} \times \mathrm{LT}^{-1} \times \mathrm{L}^{-2}} = \frac{[\text{f.e.m.}]}{\mathrm{T}^{-1}} = [\text{f.e.m.}] \times \mathrm{T}. \tag{100}$$

The electromotive force, being the circulation of the electric field, has the dimensions $\mathrm{M}^{(1/2)}$ $\mathrm{L}^{(1/2)}\,\mathrm{T}^{-1}$in the electrostatic system. Maxwell's result follows:

$$[\text{pole} = \mathrm{M}^{1/2}\mathrm{L}^{1/2}\mathrm{T}^{-1} \times \mathrm{T} = \mathrm{M}^{1/2}\mathrm{L}^{1/2} \tag{101}$$

Everett's third derivation is simply a more detailed exposition of Clausius's, expressed in terms that will enable him to draw his own conclusions.[22] Everett takes the Clausius equation expressing the equivalence between current loop and magnetic double sheet and adds the constant k_3:

[21]Everett 1882 : 432.
[22]Everett 1882 : 432-433.

Pole × Length = k_3× Current× Area

This addition allows him to conclude that the disagreement between the results of Larmor and FitzGerald on the one hand, and Clausius on the other, implies that the choices they made concerning the dimensions of their respective constants are mutually incompatible:

> [Clausius's] result disagrees with that of the two previous studies; and consequently the two equivalent hypotheses , $k_1 = 1$ $k_2 = 1$ are incompatible with the hypothesis . $k_3 = 1$ [23]

In order to obtain the same results in all cases, he says, we must either choose $[k_3] = \mathrm{T}^2\mathrm{L}^{-2}$, in which case we find the pole dimensions $\mathrm{M}^{1/2}\mathrm{L}^{1/2}$ - or take $[k_1] = [k_2] = \mathrm{T}^2\mathrm{L}^{-2}$, in which case we find $\mathrm{M}^{1/2}\mathrm{L}^{5/2}\mathrm{T}^{-2}$. The constant k_3 in the first case, and the constants k_1 and k_2 in the second case have the value $\frac{1}{v^2}$ where *v* is the quotient of the unit of electric charge in the electromagnetic system to its unit in the electrostatic system. Everett prefers the first choice, as it leads to the most commonly used formulas.

Everett doesn't stop there. He also deals with the electromagnetic system, which he considers to be by far the most practical for electromagnetic calculations. These considerations led him to introduce three other equations, involving three new factors:[24]

- the force between a current of infinite length and a parallel current of finite *length l*

$$F = k_4 \times (\text{current } 1) \times (\text{current } 2) \times 2\, l \,/\, \text{distance}, \qquad \textbf{(102)}$$

- the force between two magnetic poles :

[23] "[Clausius' result] disagrees with that of the two preceding investigations; and hence the two equivalent assumptions k_1=1, k_2= 1 are inconsistent with the assumption k_3= 1." Everett 1882: 433.
[24] Everett 1882: 433-434.

$$F = k_5 \times (\text{pole } 1) \times (\text{pole } 2) / (\text{distance})^2, \tag{103}$$

- the force between two electric charges :

$$F = k_6 \times (\text{load } 1) \times (\text{load } 2) / (\text{distance})^2. \tag{104}$$

The three factors previously discussed, k_1, k_2, and $k_3$are equal to one in the electromagnetic system. In both Maxwell's and Clausius' descriptions, $k_4$is equal to one in the electromagnetic system, and to $\frac{1}{v^2}$ in the electrostatic system,[25] while $k_6$is equal to v^2in the electromagnetic system and to one in the electrostatic system. The factor $k_5$is equal to one in the electromagnetic system (like k_4); but $k_5$is equal to v^2according to Maxwell, and to $\frac{1}{v^2}$ if we follow Clausius, as it involves the magnetic poles. From a dimensional point of view, these conclusions can be summarized as follows:

Constants	Electrostatic system		Electromagnetic system
	Maxwell	Clausius	
k_1	Digital	$L^{-2}T^2$	digital
k_2	Digital	$L^{-2}T^2$	digital
k_3	$L^{-2}T^2$	digital	digital
k_4	$L^{-2}T^2$	$L^{-2}T^2$	digital
k_5	$L^{-2}T^2$	$L^{-2}T^2$	digital
k_6	Digital	digital	$L^{-2}T^2$

Table 5.1

It will be recalled that Fourier developed dimensional considerations largely to express the invariance of physics formulas. Everett, on the contrary, insists on the conventional character of dimensional equations: while they are indeed invariant under

[25] Where v is, once again, the ratio of the electromagnetic unit to the electrostatic unit of electric charge.

change of unit within a particular system, they may differ from one system to another - where the term "system" is to be taken in the sense of given fundamental quantities and set of relations used to obtain the units of derived quantities from the units of these fundamental quantities (e.g., electrostatic system) and not simply in Fourier's sense of a given set of units for the same fundamental quantities :

> Comparing the six equations above shows how careful we must be when asserting that a dimensional relationship "must be correct in any system of units." The laws of nature that relate dissimilar quantities are laws of *proportion*, and it is only by convention that they are formulated as laws *of equality*.[26]

In other words, the laws of physics involve constants of proportionality that can have non-zero exponents of dimension. In the case of electromagnetism, these constants are either 1, v^2, or $\frac{1}{v^2}$. Even within a given system, there is a certain freedom - limited, however, by the need for equations to be invariant under changing units L, M, T - as to which constants we can afford to ignore in this way. This is an issue that will be the subject of much discussion.

5. 4 - The controversy spreads: Reactions from J. J. Thomson, J. Larmor and C. Wead

[26] Italics by Everett. "A comparison of the foregoing six equations shows how cautious we ought to be in asserting that a particular dimensional relation "must hold in every system of units." The laws of nature which connect dissimilar quantities are laws of *proportion*, and it is only by convention that they can be stated as laws of *equality*." Everett 1882 : 434. And again: "Clausius was in error in the assertion...that in every system the product of the current by the area enclosed must be *equal* to the moment of the equivalent magnet. *Proportional* would be the correct word; and the factor k_3, which remains constant as the current and area vary, does not necessarily retain the same value when we pass from one set of units of length and time to another." Everett 1882: 434.

Everett wasn't the only one to re-examine Clausius's reasoning by inserting a constant. Joseph John Thomson (1856-1940) responded to Everett's first paper (without having been privy to the discussions involving k_1, k_2 and k_3) with thoughts very similar to Everett's own.[27] However, Thomson did not share Everett's relative neutrality: he felt that Clausius had made a mistake and set out to rectify it.

According to Thomson, the Clausius starting equation,

$$\text{Pole} \times \text{Length} = \text{Current} \times (\text{Length})^2, \qquad \textbf{(105)}$$

should take into account the magnetic permeability μ, which gives :

$$\text{magnet moment} = \mu \times \text{current} \times \text{area.}^{\mathbf{28}} \qquad \textbf{(106)}$$

In Maxwell's theory, the force between two magnetic poles is inversely proportional to the magnetic permeability of the medium, whereas the force between a current and a pole is independent of the medium.[29] In the electrostatic system, the current has dimensions $M^{1/2}L^{3/2}T^{-2}$, and μ has dimensions . $L^{-2}T^{2}$ [30] The formula for the dimensions of a pole is :

$$[\text{pole}]\ , = \left[\frac{\mu \times \text{courant}\ \times \text{aire}}{\text{longueur}}\right] = \frac{T^2L^{-2} \times M^{1/2}L^{3/2}T^{-2}\ \text{x}\ L^2}{L} = M^{1/2}L^{1/2} \qquad \textbf{(107)}$$

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[27] Thomson J.J. 1882a: 427-429, sent May 16th 1882. Everett makes no reference to Thomson's remarks in his second article, which seems to have been published at the same time.

[28] Thomson J.J. 1882a: 427.

[29] Thomson J.J. 1882a: 429.

[30] Thomson offers a demonstration of this, but unfortunately his reasoning seems to be circular. Having in fact already obtained the dimensions of a pole as equal to $M^{(1/2)\ L1/2}$, by a different reasoning that we report below, he deduces those of the magnetic induction m/r^2, i.e. $M^{(1/2)\ L-3/2}$. The dimensions of the magnetic force in the electrostatic system are $L^{(1/2)\ M}(^{1/2)\ T-2}$, as Maxwell states in his treatise, vol.2, 242. Permeability, being the quotient of magnetic induction and magnetic force, therefore has the following dimensions: $M^{(1/2)\ L-3/2}$/ $L^{(1/2)\ M(1/2)\ T-2}$= $L^{(-2)\ T(2)\ (}$Thomson J.J. 1882a: 428). Note, however, that these dimensions can be obtained as follows: in the electrostatic system, the force between two currents has the dimensions $[\mu i i']$ or $\left[\mu\frac{qq'}{tt'}\right]$ and the force between two charges $\left[\frac{qq'}{d^2}\right]$. We therefore have $\left[\mu\frac{qq'}{tt'}\right]=\left[\frac{qq'}{d^2}\right]$ from which: $[\mu]=T^2L^{-2}$

which is Maxwell's result.[31] Thomson thus rediscovers what Everett thought necessary to derive Maxwell's result from Clausius' reasoning: take $k_3 = \frac{1}{v^2}$, instead of . $k_3 = 1$

In the rest of his article, Thomson presents another demonstration of the dimensions of a pole in the electrostatic system. For this, he considers the work done by the force due to a current *i* in a long straight wire, on a pole of intensity *m* located at a distance *r* from the wire, when the pole describes a complete circle in a plane perpendicular to the wire :

$$\text{work} = \frac{2i}{r} m \times 2\pi r = 4\pi m i. \qquad (108)$$

Consequently, the product mi must have the dimensions of a work, ML^2T^{-2} . Since the current has electrostatic dimensions $M^{1/2}L^{3/2}T^{-2}$, Thomson obtains for the dimensions of a pole :

$$[m] = \frac{ML^2T^{-2}}{[i]} = \frac{ML^2T^{-2}}{M^{1/2}L^{3/2}T^{-2}} = L^{1/2}M^{1/2}. \qquad (109)$$

The criticism that Clausius levelled at Maxwell can certainly be levelled at this demonstration: it is based on a relationship expressing electrodynamic phenomena.

In the same issue of *Philosophical magazine*, Larmor criticizes Clausius's approach for reasons that differ from those of Thomson and Everett.[32] He considers that some relationships are more fundamental, and above all more natural, than others. In his view, the formula on which Clausius is based,

$$\text{Magnet moment} = \text{current} \times \text{area},^{33} \qquad (110)$$

[31] Thomson J.J. 1882a: 429.
[32] Larmor 1882: 429-430.
[33] Larmor 1882: 429.

involves the electromagnetic system, and must be accorded a lower status than others. The reason is that it seems more metaphorical than real to him:

> The somewhat abstruse fact that the magnetic action of a small circuit in which a current flows can be represented as being due to two fictitious magnetic poles, doesn't seem likely to override the natural statement of the only fundamental relationship by which natural poles play a role in the theory of electricity. This relationship is, of course, electromagnetic by its very nature.
> It could be argued that the system to which we object has built the theory on electrical foundations, in that its fundamental magnet is a small electric current. But, in response to this argument, it's the existence of real magnets that introduces the idea of pole.[34]

Note that the reason Larmor dislikes Clausius's relation is precisely the reason Clausius considered it a legitimate choice: the fact that it expresses an equivalence between two different situations, rather than an action between two entities.

Larmor gives greater theoretical legitimacy to three other formulas, two of which he gives explicitly:

$$\text{(1) (Pole)}^2 \times L^{-2} = \text{Force,} \tag{111}$$
$$\text{(2) Current} \times L \times \text{Pole} \times L^{-2} = \text{Force.} \tag{112}$$
[35]

For Larmor, these relations are more fundamental in the sense that, to obtain the Clausius relation, they must necessarily be taken into account, however tacitly. The first is the expression of the force between two magnetic poles as a function of their intensity and distance, and Larmor describes it as "the purely magnetic starting point of the electromagnetic system."[36]

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[34] "The somewhat recondite fact that the magnetic action of a small circuit carrying a current may be represented as due to two fictitious magnetic poles, does not seem to possess any claim to supplant the natural statement of the only fundamental relation which makes natural poles play a part in electric theory at all. That relation is, of course, itself in its very nature electromagnetic. It might be said that the system objected to built up the theory from electric foundations, inasmuch as its fundamental magnet is a small electric current. But, in reply, it is the existence of actual magnets which introduces the idea of pole at all" Larmor 1882: 430.
[35] Larmor 1882: 430.
[36] That is, "the purely magnetic starting-point of the electromagnetic system," Larmor 1882: 430.

The second formula expresses the force between a magnet and an electric current, which he used in the demonstration reported by Everett. For Larmor, this relationship is the only one that can legitimately connect the notion of pole to that of current:

> The only natural fact linking pole to current is Oersted's phenomenon (developed by Ampère) of the existence of a mutual force between them: this is expressed by ([27]); and we must therefore deduce from this relationship the dimensions of a pole previously established.[37]

As for preferring relation (25) to relation (27), Larmor adds:

> [This choice], while permissible if consistently adhered to, does not seem natural - if specifying a magnetic pole in an electrostatic system has any significance at all.[38]

It's worth noting that, for Larmor, the electrostatic system and electrostatic phenomena themselves must be considered more fundamental than their electromagnetic counterparts. He therefore establishes a hierarchy not only between equations, but also between systems of units.

As for the result we obtain for the dimensions of a magnetic pole if we follow the principles he defends, Larmor simply mentions that it is the formula given by Maxwell, $L^{1/2}M^{1/2}$. His reasoning is none other than that reported by Everett in his second article, which we have already discussed.[39]

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[37] "The only simple fact of nature which connects pole with current is Oersted's phenomenon (as developed by Ampère) of the existence of a mutual force between them: this is expressed by (27); and hence by this relation must we deduce the dimensions of pole already found." Larmor 1882: 430.

[38] "This, though allowable if consistently adhered to, does not seem to be natural - if indeed the specification of a magnetic pole on an electrostatic system is a matter of importance at all." Larmor 1882: 430.

[39] Larmor 1882: 430.

The controversy continued with a certain Charles Kasson Wead (1848-?), professor of physics at the University of Michigan,[40] , who also sided with the formula obtained by Maxwell rather than that of Clausius. [41] His reasons seem to be primarily practical:

> If P [the dimensions of a pole] is modified, three other quantities among the twelve discussed by Maxwell must have their dimensions modified, and confusion would thus be introduced into his system, which is based on fifteen equations, in each of which the second member is a simple mechanical quantity, such as work, time, etc., and the first member is a simple mechanical quantity. Until it has been clearly shown how this system will be affected by the proposed change, and why the new expression is preferable to the old one, which has remained unchallenged for nearly twenty years, isn't it clearly preferable to write $P_e= M^{(1/2)\ L1/2}$? [dimension of a magnetic pole in the electrostatic system]. This is not simply a question of accuracy, but of consistency, simplicity and usefulness, and for all these reasons Maxwell's expression seems to us to deserve preference.[42]

Wead offers a detailed discussion of the issue. He uses the following symbols for quantities or rather their dimensions:

C: current intensity
I: magnetic field strength
μ or *PL*: moment of a magnet
P: magnetic pole current
Q: quantity of electricity

He uses the indices e and m to designate the system of units in question.[43] He begins by deriving the result reported by Everett concerning the dimensions of a pole according to Clausius: . $M^{1/2}L^{5/2}T^{-2}$ [44] His demonstration is very brief; he gives the equation :[45]

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[40] Burke 1906: 255-256.
[41] Wead 1882: 530-533.
[42] "If P be changed, three other quantities of the twelve that Maxwell discusses must have their dimensions changed, and confusion would be introduced into his system, that is based on fifteen equations, in each of which the second member is some simple mechanical quantity, as work, time, &c. Until it has been clearly shown how this system will be affected by the proposed change, and why the new expression is to be preferred to the older one, that has been "unimpeached" for some twenty years, is it not clearly better to write $P_e= M^{(1/2)}$ $^{L1/2}$? It is not a question merely of correctness, but of consistency, simplicity, and usefulness; and on all these grounds Maxwell's expression seems to the writer to deserve the preference." Wead 1882: 533.
[43] Wead 1882: 530.
[44] Wead 1882: 530-531.
[45] Equation which amounts to Current × $(\text{Length})^2$= Pole × Length.

$$\mathrm{C} \times \mathrm{L}^2 = \mathrm{P} \times \mathrm{L} \qquad \textbf{(113)}$$

and obtains $\mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-2}$ for the pole dimensions by assigning electrostatic dimensions to the current. He also offers a new derivation of Maxwell's result. This derivation, by Hermann Herwig, then Professor of Physics at Darmstadt, is based on the torque exerted on a magnet by a magnetic field. The torque is equal to the field strength multiplied by the magnet's magnetic moment:

$$\text{couple} = I_e \times \mu_e \qquad \textbf{(114)}$$

$$\Rightarrow \qquad \mu_e = \frac{\text{couple}}{I_e} = \frac{\mathrm{ML}^2\mathrm{T}^{-2}}{\mathrm{M}^{1/2}\mathrm{L}^{1/2}\mathrm{T}^{-2}} = \mathrm{M}^{1/2}\mathrm{L}^{3/2} \qquad \textbf{(115)}$$

And since $\mu_e = P_e\mathrm{L}$, we obtain :

$$P_e = \frac{\mu_e}{\mathrm{L}} = \mathrm{M}^{1/2}\mathrm{L}^{3/2} \times \frac{1}{\mathrm{L}} = \mathrm{M}^{1/2}\mathrm{L}^{1/2}. \qquad \textbf{( )116}$$

Like Larmor, Wead prefers this derivation to that of Clausius because of the status he gives to the parallel drawn between current and magnet:

> [According to Clausius] $PL = \mu$, i.e. PL must represent a magnet; therefore $C \times L^2$ is taken to equal a magnet. But the passage of current generally produces effects, such as the movement of a galvanometer needle, which we explain more naturally by saying that the circular current produces a magnetic field at its center, than by saying that the current is, or produces, or even is equivalent to, a magnet. So Herwig's demonstration, in which a magnet placed in a field undergoes a torque, is more in line with the usual way of seeing things than is Clausius', and it's the one used to obtain the electromagnetic system.... It should be noted that, in all discussions other than that of Clausius, the P pole of the magnet is introduced into the field. Clausius produces a field with the P pole. [46]

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[46] "PL = μ, that is, PL must represent a magnet; consequently C × $L^2$is put equal to a magnet. But the passage of the current ordinarily produces effects, such as the movement of a galvanometer-needle, which we explain more naturally by saying that the circular current produces a magnetic field at its center, than by saying that the current is, or makes, or even is equivalent to, a magnet. Herwig's derivation, therefore, in which a magnet placed in a field experiences a couple, conforms to the ordinary way of thinking better than the way of Clausius, and is the way used in the derivation of the electromagnetic system.... It may be noted that, in all discussions except that of Clausius, the magnet pole P is introduced into the field. Clausius produces a field by the pole." Wead 1882: 531.

Like Larmor, Wead considers "that to write $\mathrm{C} \times \mathrm{L}^2 = \mathrm{P} \times \mathrm{L}$ in any coherent system is to assume what is in question to be true."[47] And he establishes a hierarchy between the equations used in physics. The most fundamental equations of magnetism are those used by Gauss in the absolute measurement of the earth's magnetic field (for the horizontal oscillation of a suspended magnet and for the direction taken by a compass subjected to both the earth's field and the magnet's field):[48]

$$\mu H = \frac{\pi K}{t^2(1+\theta)} \qquad\qquad \frac{\mu}{H} = \frac{1}{2}\frac{\gamma^5 \tan\varphi - \gamma'^5 \tan\varphi'}{\gamma^2 - \gamma'^2} \tag{117}$$

with	K: moment of inertia	dimensions :	ML^2
	t : time		T
	γ and γ': lengths		L
	$\tan\varphi$ and θ :		numerical values

For dimensional formulae - replacing H by I - we obtain the magnetic field :

$$[\mu\, I] = \frac{\mathrm{ML}^2}{\mathrm{T}^2} \qquad\qquad \left[\frac{\mu}{I}\right] = \frac{\mathrm{L}^5}{\mathrm{L}^2} = \mathrm{L}^3 \tag{118}$$

From this we obtain for the dimensions of μ : $\mathrm{M}^{1/2}\mathrm{L}^{5/2}\mathrm{T}^{-1}$, and for those of I: $\mathrm{M}^{1/2}\mathrm{L}^{-1/2}\mathrm{T}^{-1}$. The dimension formulas (33), being the direct consequence of the two equations (32) above, deserve according to Wead to be considered correct in any system of units. However, this is not the case. In the system proposed by Maxwell, the first formula is correct (and Herwig uses it explicitly), but not the second; whereas in the system proposed by Clausius, the opposite is true. Clausius doesn't use the second formula per se, but his equations imply it: [49]

[47] "Ro write 'C × L^2= P × L in any consistent system is simply begging the whole question." Wead 1882: 532.
[48] Wead 1882: 532. Wead uses M to mean moment; we useµ to avoid possible confusion with the fundamental quantity of mass.
[49] Wead 1882: 533.

$$C\,\mathrm{L}^2 = \mu\,,\ \frac{C}{\mathrm{L}} \times \mathrm{L}^3 = \mu\,,\ \frac{C}{\mathrm{L}} = I \quad \Rightarrow \quad \frac{\mu}{I} = \mathrm{L}^3 \tag{119}$$

Wead takes a second example, the equation: $I = k\dfrac{2\pi r^2 C}{\left(l^2 + d^2\right)^{3/2}}$ for the magnetic field created by a current loop.

with: r: radius of a circular electric circuit — dimension: L
d: distance from the center of the circle, taken from the normal to the plane passing through the center — dimension: L
l : ?? — dimension: L

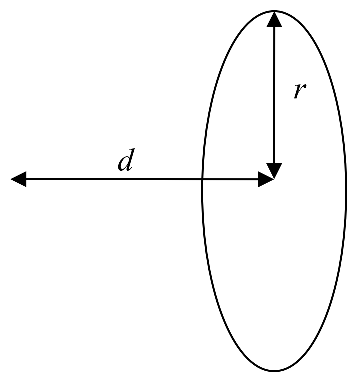

Figure 5.2

Ordinarily, we assume k equals 1. According to Wead, we could just as well give ourselves C and I at the same time, in which case k would most certainly have dimensions.

Similarly, to obtain the dimensions of a pole and a magnetic moment from I, we use :

$$I = \frac{kP}{l^2} \tag{120}$$

and

$$\mu I = k' \times \text{ torque} \tag{121}$$

Here again, the numerical constant k is taken by Clausius to be equal to one and dimensionless, and thus determines which dimensions k' must take on for (36) to be satisfied. In contrast, Herwig takes k' to be equal to one. Wead adds that if we take both k and k' to be equal to one, we obtain the same dimensions for μ and I as in the electromagnetic system.[50]

Furthermore, Wead compares the relationships by which various other quantities are obtained in different systems (and the order in which they are obtained):

Electromagnetic system :

$$\frac{PP}{\mathrm{L}^2} = F \qquad P\,\mathrm{L} = \mu\,; \quad I\,\mu = \text{torque}; \frac{C\,\mathrm{L}}{\mathrm{L}^2} = I\ ;\ Q = C\,\mathrm{T} \tag{122}$$

So first we get P, then μ, then I, C, and finally Q.

Electrostatic system :

$$\frac{QQ}{\mathrm{L}^2} = F, \qquad \frac{Q}{\mathrm{T}} = C, \qquad \frac{C\,\mathrm{L}}{\mathrm{L}^2} = I, \qquad I\,\mu = \text{torque}, \quad \mu = P\,\mathrm{L}\,. \tag{123}$$

So we get Q first, then C, then I, μ, and finally P.

Electrostatic system (Clausius) :

$$\frac{QQ}{\mathrm{L}^2} = F, \qquad \frac{Q}{\mathrm{T}} = C, \qquad C\,\mathrm{L}^2 = \mu = P\,\mathrm{L}. \tag{124}$$

As Wead says, the first two systems differ in the way they obtain their fundamental quantity, i.e. $P_m$and Q_e. Once these quantities have been obtained, the same relations are used in both systems to obtain the other physical quantities, albeit in reverse order. But some of these relations differ from those used in system (39), notably the Clausius relation $C\,\mathrm{L}^2 = \mu$. Wead considers this difference between systems (38) and (39) (it is not found between (37) and (38)) to be an argument against the system advocated by Clausius.

[50] Wead 1882: 533.

Wead then uses these three sets of formulas to establish dimensional relationships between Q and P in each of the systems. He thus obtains :[51]

$$Q_m = \frac{\mathrm{ML^2T^{-1}}}{P_m} \tag{125}$$

$$P_e = \frac{\mathrm{ML^2T^{-1}}}{Q_e} \tag{126}$$

$$P_e = Q_e \mathrm{L\,T^{-1}} \tag{127}$$

Since P_e and Q_m play the same role in their respective unit systems, Wead expects a certain symmetry between the corresponding formulas. We can see that (40) and (41) exhibit such symmetry, but not (40) and (42). It therefore seems natural to prefer (41) to (42). [52]

In sum, although Wead's considerations differ from those of the previous authors in detail, his concerns are similar to theirs in at least two respects. Firstly, like Everett in his second article, he uses the idea of additional dimensional factors. Secondly, Wead shares with Larmor the idea that the ambiguity of dimensions in a given system (electrostatic or electromagnetic) is linked to the hierarchy we admit between the equations of the theory: certain equations are deemed more "natural" or more "fundamental." However, as far as he's concerned, this judgment is purely subjective, and so the Clausius relation is not false. Wead even goes so far as to say:

> There's almost as much arbitrariness in the way we obtain the dimensions of physical quantities, as in the order in which we multiply several numbers together.[53]

For Wead, preferring Maxwell's formula to Clausius' is purely a matter of convenience.

[51] Wead 1882: 532.
[52] Wead 1882: 532.
[53] "In deducing the dimensions of physical quantities, there is much that is as arbitrary as the order in which several numbers shall be multiplied together." Wead 1882: 533.

In September of the same year, J.J. Thomson published another article[54] which undoubtedly helped to clarify the situation. Not that Thomson's article reflected any profoundly different concerns from those he had expressed a few months earlier. It simply takes a step back from Clausius' reasoning. Remember that in May, Thomson's approach was to insert a medium-dependent constant, magnetic permeability, into Clausius' relationship between current loop and magnetic moment. In the September article, the same preoccupation with medium-dependent constants is repeated. However, Thomson's starting point is no longer the formula that translates the equivalence between a magnetic envelope and a circuit, but the equation translating the magnetic force between two poles (as well as its electrostatic counterpart). In particular, he criticized Clausius for thinking that the attraction between two magnetic poles could have dimensions other than those of a force. Indeed, if we take the electrostatic dimensions of a pole according to Clausius, $[m_S] = \mathrm{M}^{1/2}\mathrm{L}^{5/2}\mathrm{T}^{-2}$, we obtain :

$$[F] = \left[\frac{m_S m'_S}{r^2}\right] = \frac{\left(\mathrm{M}^{1/2}\mathrm{L}^{5/2}\mathrm{T}^{-2}\right)^2}{\mathrm{L}^2} = \frac{\mathrm{ML}^5\mathrm{T}^{-4}}{\mathrm{L}^2} = \mathrm{ML}^3\mathrm{T}^{-4} \qquad \textbf{(128)}$$

where m and m' represent pole intensities.

Thomson questions the first of these equalities. He argues that, since the attraction between two poles is clearly a force, if $\frac{mm'}{r^2}$ does not have the dimensions of a force (MLT^{-2}), this necessarily means that this formula is incomplete and that a factor of appropriate dimensions must be inserted before it can be said to be equal to the force. Although Thomson does not specify it, it goes without saying that this factor must be

[54] Thomson J.J. 1882b: 225-226.

$\frac{\mathrm{M\,L^3T^{-4}}}{\mathrm{M\,L\,T^{-2}}} = \mathrm{L^2T^{-2}}$. Thomson notes that it is an advantage of Maxwell's system that it does not require the artificial introduction of such additional factors, as homogeneity considerations are already satisfied by the constants characterizing the medium:

> These factors appear naturally through symbols that represent a physical property of the body or environment. [55]

Thomson begins by illustrating this point in the case of the electrostatic force between two charged particles, which is expressed by the formula $\frac{ee'}{\kappa r^2}$ - where *e* represents the charge of the particles and *r* their distance. *κ* is the dielectric permittivity of the medium in which the charges are located. Thomson considers the dimensions of this expression in both the electrostatic and electromagnetic systems:

System	e	κ	$\frac{ee'}{\kappa r^2}$
Electrostatic	$\mathrm{M^{1/2}L^{3/2}T^{-1}}$	no dim.	$\frac{(\mathrm{M^{1/2}L^{3/2}T^{-1}})^2}{\mathrm{L^2}} = \mathrm{MLT^{-2}}$
Electromagnetic	$\mathrm{M^{1/2}L^{1/2}}$	$\mathrm{L^{-2}T^2}$	$\frac{(\mathrm{M^{1/2}L^{1/2}})^2}{\mathrm{L^{-2}T^2L^2}} = \mathrm{MLT^{-2}}$

Table 5.2

[55] "These factors introduce themselves naturally through symbols representing some physical property of the body or medium." Thomson J.J. 1882b: 226.

So in both systems, the expression has the same dimensions, those of a force, and it's through the κ factor that this is made possible.

After this preliminary illustration, Thomson turns to the more controversial case of magnetic force. He treats it in exactly the same way. The expression for the force this time is $\frac{mm'}{\mu r^2}$ (where m is the value of a magnetic pole and μ represents the permeability of the medium). We now obtain :

System	m	μ [56]	$\frac{mm'}{\mu r^2}$
Electrostatic	$M^{1/2}L^{1/2}$	$L^{-2}T^2$	$\frac{(M^{1/2}L^{1/2})^2}{L^{-2}T^2L^2} = MLT^{-2}$
Electromagnetic	$M^{1/2}L^{3/2}T^{-1}$	no dim.	$\frac{(M^{1/2}L^{3/2}T^{-1})^2}{L^2} = MLT^{-2}$

Table 5.3

(Note the symmetry between the dimensional formulas for m and e in both systems). [57]

Thomson concludes with the following generalization:

> The reader can easily verify the assertion that in Maxwell's system every equation is correct as it stands, and therefore whenever we have a purely dynamic effect, its expression will have the same dimensions in both systems of units.[58]

5. 5 - The end of the controversy: Oliver Lodge's intervention

[56] The dimensions of μ in the electrostatic system are the inverse of the "factor" dimensions derived above, because μ is in the denominator.
[57] Thomson J.J. 1882b: 226.
[58] "the reader can easily verify the statement that in Maxwell's system every equation is true as it stands; and consequently, whenever we have a purely dynamic effect, the expression for it will be of the same dimensions in both systems of units."
Thomson J.J. 1882b: 226.

The controversy came to an end with an article by Maxwellian experimenter Oliver Joseph Lodge (1851-1941),[59] followed by an article by E.B. Sargant, who in essence merely expressed his agreement with Lodge's (probably Edmund Beale Sargant (1855-1938), but this is not certain; the latter, educated at Cambridge, went on to distinguish himself through his contributions to education, notably in South Africa).[60] Lodge's approach is particularly interesting in that he offers a synthetic view of the debates just discussed, and attributes the disagreements to a difference in the scientific culture of the two parties, linked to their national affiliation:

> The discussion that took place in the pages of your newspaper on the dimensions of a magnetic pole serves to illustrate the divergence of thought between those in this country educated in the field of electricity under the influence of Faraday and Maxwell, and the continental philosophers so eminently represented by Professor Clausius. From one point of view, it can be said that the whole discussion stems from a simple misunderstanding that arises from the neglect of a factor in one of the members of an equation.... But on the other hand, it is the natural and inevitable consequence of the two approaches to the question: the English point of view, in which the environment is recognized as the active agent and is constantly present in both minds and formulas; and the continental point of view, in which the environment is perceived as so much empty space and rendered as such in formulas.[61]

The beginning of Lodge's discussion is very similar to that of J.J. Thomson's above.[62] Lodge begins by establishing the validity of taking "specific inductive power" into account

[59] Lodge 1882: 357-365.
[60] Sargant 1882: 395-396.
[61] "The discussion which has been carried on in your pages respecting the dimensions of a magnetic pole serves to illustrate the divergence of thought between those in this country who have been brought up, electrically, under Faraday and Maxwell, and the continental philosophers so eminently represented by Prof. Clausius. From one point of view the discussion may be said to have been roused by a simple mare's nest constructed by dropping a factor out of one side of an equation...; but from another point of view it is the natural and inevitable consequence of the different aspects from which these matters can be regarded : the English standpoint, in which the medium is recognized as the active agent, and is continually present both in the mind and in the formulæ; and the continental standpoint, from which the medium is perceived as so much empty space, and is taken account as such in the formulæ." Lodge 1882: 357.
[62] Thomson J.J. 1882b: 226.

in the equation that gives the force between two electric charges. He even takes the trouble to recall the stages in the discovery of this quantity.

Lodge explains that Coulomb, on the continent, first established the law that bears his name as proportionality:[63]

$$\text{force between two loads} \propto \frac{ee'}{r^2}\,. \tag{129}$$

Faraday, in England, then established that this relationship is only valid if the medium in which the charges are found does not change, and he transformed the proportionality into equality thanks to a K factor depending on the medium in which the charges are found :

$$F = \frac{ee'}{K\,r^2}\,. \tag{130}$$

Lodge then performs dimensional reasoning similar to that of Thomson. But instead of giving himself the dimensions of the various quantities, he pretends to have to determine those of e and K. He therefore assumes that $\frac{e^2}{K}$ must have as its dimensions those of $F\,r^2$.

A From this equation alone, however, the dimensions of e and K cannot be determined independently, and Lodge therefore expresses one as a function of the other:[64]

$$[e] = \mathrm{L}\,[KF]^{1/2}\,. \tag{131}$$

He insists that this formula is true regardless of the system under consideration. The next arbitrary step is to take K as dimensionless - equal to 1 in air - to obtain the electrostatic system.

[63] Lodge 1882: 357-358.
[64] Lodge 1882: 358.

Having illustrated his reasoning on the unanimous case, Lodge now applies it to the problem of magnetic poles.[65] Here again, it is necessary to take into account a quantity μ characterizing the medium in order to obtain the force between two magnetic poles:

$$F = \frac{mm'}{\mu r^2}. \tag{132}$$

We can then express the dimensions of m in terms of those of the other quantities:

$$[m] = \mathrm{L}\,[\mu F]^{1/2} \tag{133}$$

or :

$$\frac{[e]}{[\sqrt{K}]} = \frac{[m]}{[\sqrt{\mu}]} = \mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-1}. \tag{134}$$[66]

Here again, arbitrariness intervenes in the definition of the magnetic system by making μ a dimensionless quantity equal to 1 in air.

Lodge insists that, despite the perfect symmetry between the electrostatic and magnetic cases, the arbitrary definitions of the dimensions (or units) of *e* in the electrostatic system, and of *m* in the electromagnetic system, are totally independent of each other: the former depend solely on the choice of *K* and the latter on that of μ :

> The arbitrary definition of *e* and the arbitrary definition of *m* are entirely independent of each other in their origin; not that they are unrelated, but their relationship will have to be the subject of future experimental research.[67]

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[65] Lodge 1882: 358.
[66] Lodge 1882: 358.
[67] "The arbitrary definition of *e* and the arbitrary definition of *m*, are in their origin utterly independent; not that they are unrelated, but their relation must be a matter for future and experimental investigation." Lodge 1882: 358.

It's only by taking into account another experimental fact that we obtain a relationship between m and e, namely the one on which Clausius bases his reasoning: the fact that an electric current produces a magnetic field, and can therefore be considered equivalent to a magnet. The mathematical expression chosen by Lodge to express this is the same as that of J.J. Thomson: [68]

Magnet moment $\propto$ current intensity [69] $\times$ circuit area.

The question is whether this proportional relationship can be taken as an equality, or whether, on the contrary, the effect of the medium in which the circuits or magnets are immersed must be taken into account.[70] Lodge insists that this question can only be answered by experiment. Pending such results, he proposes a number of experimental projects.

Lodge first considers a solenoid and a magnet such that they produce the same external magnetic field in air. The question is whether the external magnetic field changes if the two arrangements are immersed in the same medium of magnetic permeability μ

[68] Lodge 1882: 359.

[69] There is one difference between Lodge and the previous authors that is worth mentioning, although it does not affect the rest of his reasoning and the question at hand. Lodge proposes several different definitions of the unit of current: "The unit of current most simply and directly applicable to these electromagnetic phenomena is not the old electrostatic unit at all, but a new unit which may be defined in many ways - as, for instance, these : The electromagnetic unit of current is that which produces unit magnetic potential at a point whence its circuit subtends unit solid angle; It is also that which produces unit magnetic intensity, in a given direction, at a point whence the solid angle subtended by its circuit is changing at unit rate, per unit displacement, in that direction; And, again, it is that current which when flowing round a contour of unit area is equivalent to a magnet of unit moment, all these statements being derived directly from the unit magnetic pole thus : Unit magnetic potential is defined to exist wherever a solitary and stationary unit pole would possess unit energy; Unit magnetic intensity exists wherever unit pole would experience unit force; and Unit magnetic moment is that possessed by two unit poles of opposite sign rigidly connected by a bar of unit length." We can see that these various definitions are based on the idea of equivalence between an electric current and a magnet, expressed in different ways. This diversity of definitions may come as a surprise. It is due to Lodge's desire to be as comprehensive as possible, given that his article is intended as a synthesis. Implicitly, Lodge believes that these definitions are equivalent and that they all lead to the same dimensions for a given physical quantity. (Lodge 1882: 359).

[70] Lodge 1882: 360.

different from one, as Maxwell's theory indicates. Lodge imagines wrapping the solenoid in a non-magnetizable material to separate the outer and inner regions of the solenoid. Then he plans to immerse both arrangements in a magnetic medium - the inside of the solenoid itself still being air. Lodge asserts that in this case, following Maxwell's theory, magnet and solenoid will continue to produce the same external magnetic field, as they did when both were in air. The field strength at any point in this external region, he says, will now be the value it had in air, divided by μ, in both cases.

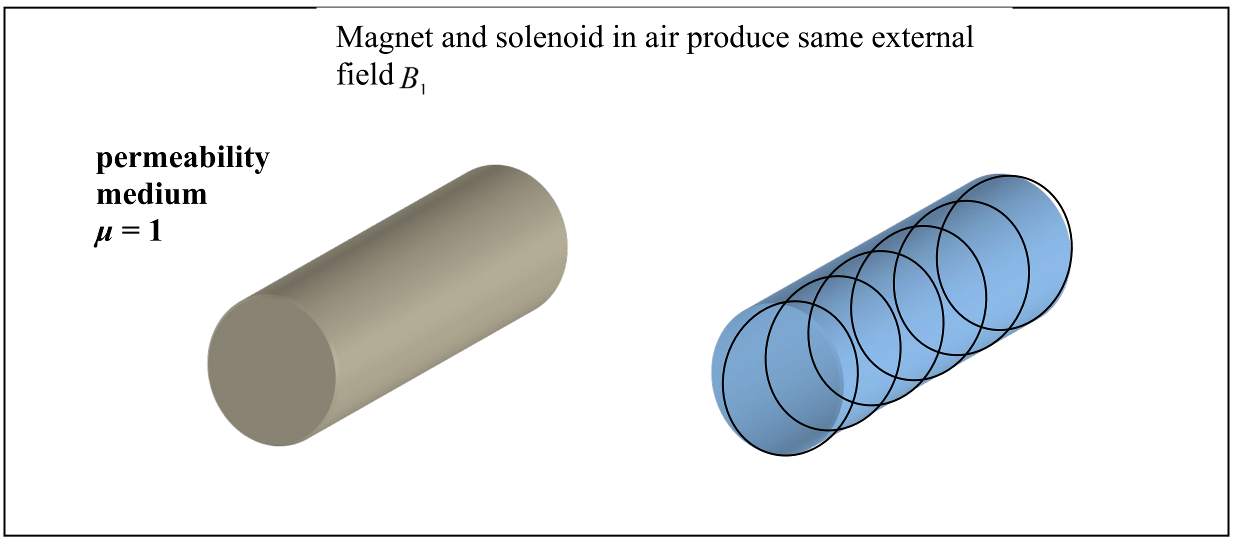

Figure 5.3

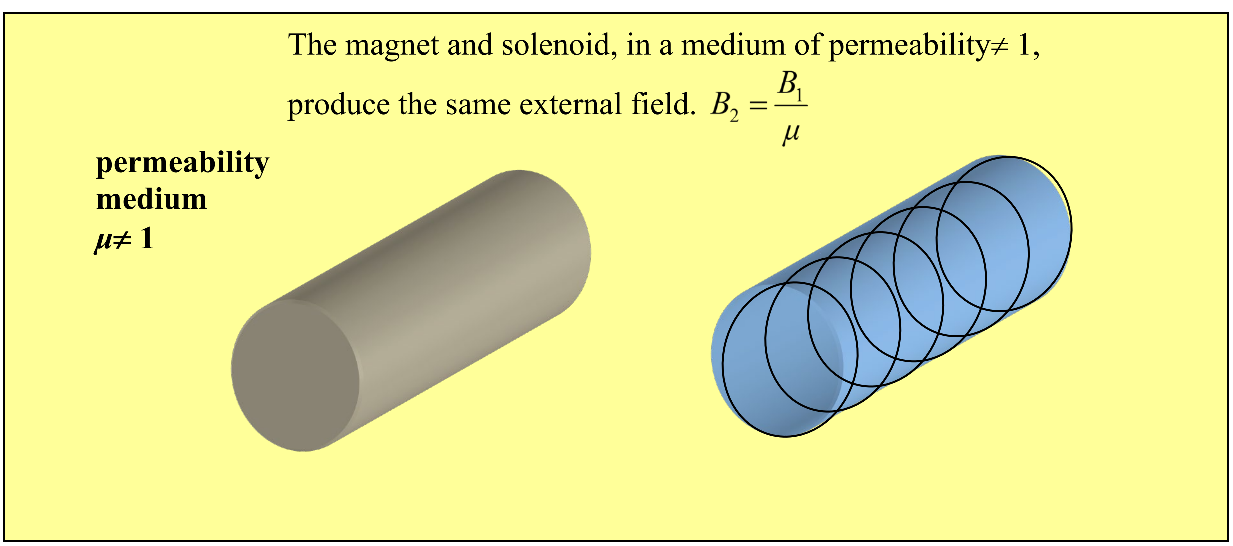

Figure 5.4

Lodge then imagines removing the envelope surrounding the solenoid while leaving it in the same medium. He asserts that in this case, following Maxwell's theory, the field produced by the solenoid will be more intense than that produced by the magnet, by a factor μ. In fact, the magnet will always produce a magnetic field of intensity $\frac{1}{\mu}$ that it produced in air, whereas the magnetic field produced by the solenoid will be μ times more intense than when the envelope was present; this field will therefore be the same as in air.[71]

The magnet and solenoid, in a medium of permeability≠ 1, which is also present inside the solenoid, no longer produce the same external field:

-the magnet produces $B_2 = \frac{B_1}{\mu}$

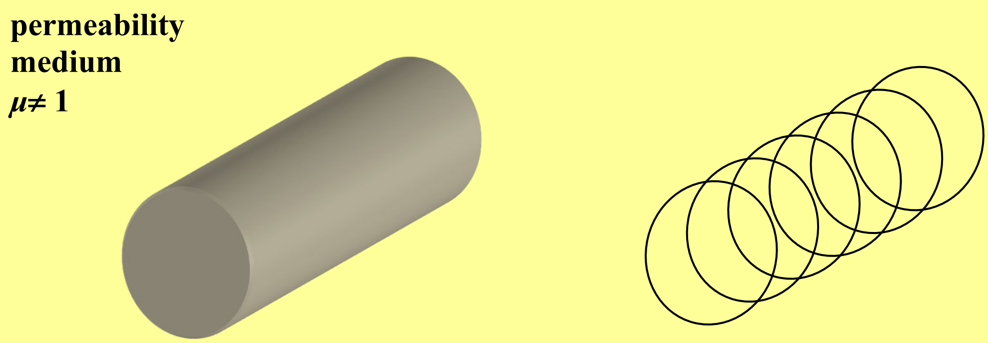

Figure 5.5

Lodge insists on the need to carry out these experiments and mentions that Thomson suggested experiments in this respect that Sargant intended to implement.[72]

Lodge didn't stop there. He imagines wrapping the solenoid again, as before, while it is still fully immersed in the magnetizable medium, and removing the two arrangements

[71] Lodge 1882: 361.
[72] Lodge 1882: 361-362.

from this medium, placing them in air. The medium with a permeability other than one is thus trapped in the region inside the solenoid. The magnetic field produced by the solenoid, Lodge asserts, will remain μ times stronger than that produced by the magnet according to Maxwell's theory.[73]

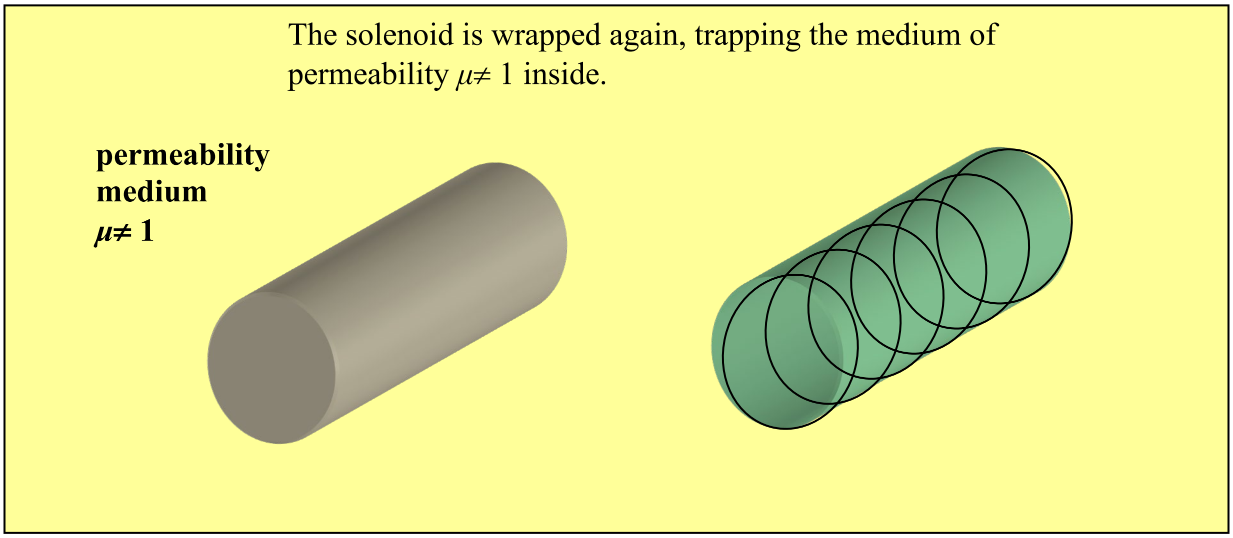

Figure 5.6

[73] Lodge 1882: 362.

The magnet and solenoid containing the fluid of permeability $\mu \neq 1$ are placed in air.
The magnet and solenoid do not produce the same external field:
-the magnet produces $B_{(1)}$
the solenoid, with μ times greater internal permeability, produces $B_3 = \mu B_1$

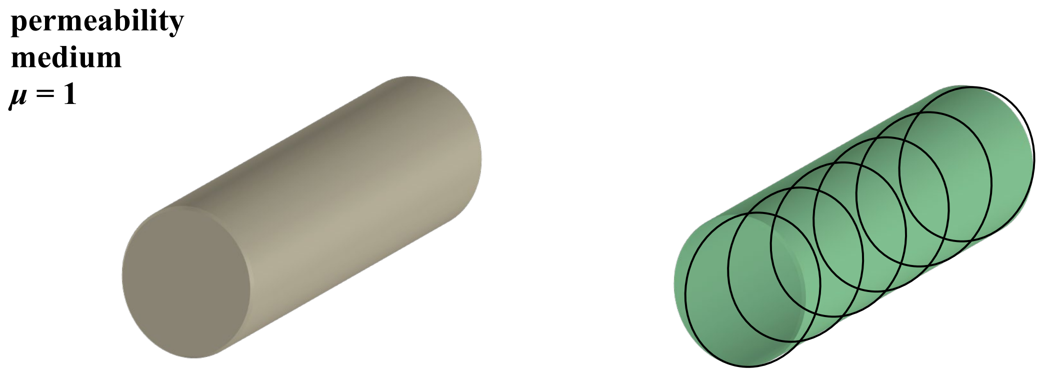

Figure 5.7

Provided that these predictions are correct, Lodge deduces the following rule:

> The magnetic moment of a circuit is equal to its current intensity multiplied by its effective area and further multiplied by the magnetic inductive capacity (or permeability) of the medium in the region inside the contour (which for a simple planar circuit is simply an envelope), but is entirely independent of the magnetic properties of the rest of the medium outside.[74]

Let be : Magnetic moment $=$ $\mu_{\text{inside}} \times$ current intensity$\times$ circuit area. **(135)**

Lodge immediately translates this statement into dimensional terms, giving the equation :

$$[m]\mathrm{L} = [\mu e]\mathrm{T}^{-1}\mathrm{L}^2 \qquad \textbf{(136)}$$

$$[m] = \mathrm{L}[\mu e]\mathrm{T}^{-1} \qquad \textbf{(137)}$$

He then uses the relation (49) we saw earlier to replace $[m]$:

[74] "The magnetic moment of a circuit is equal to the strength of its current multiplied by its effective area and again multiplied by the magnetic inductive capacity (or permeability) of the medium in the interior of the region enclosed by the contour (which region for a simple plane circuit is a mere shell), but is wholly independent of the magnetic properties of all the rest of the surrounding medium." Lodge 1882: 362.

$$\left[eK^{-1/2}\right] \equiv \left[m\,\mu^{-1/2}\right] \qquad \textbf{(138)}$$
$$\left[m\right] \equiv \left[eK^{-1/2}\mu^{1/2}\right] \qquad \textbf{(139)}$$

and therefore :

$$\left[eK^{-1/2}\mu^{1/2}\right] \equiv \mathrm{L}\left[\mu\, e\right]\mathrm{T}^{-1} \qquad \textbf{(140)}$$
$$\left[K^{-1/2}\mu^{-1/2}\right] \equiv \mathrm{L\,T}^{-1} \qquad \textbf{(141)}$$
$$\left[K\,\mu\right] \equiv \left(\mathrm{L\,T}^{-1}\right)^{2} \equiv \frac{1}{v^{2}} \qquad \textbf{(142)}$$

In addition :

$$\left[\frac{e}{m}\right] \equiv \left[\frac{e}{e\,K^{-1/2}\mu^{1/2}}\right] \equiv \left[\frac{K}{\mu}\right]^{1/2} \qquad \textbf{(143)}$$

and, since $\left[K\,\mu\right] \equiv \frac{1}{v^{2}}$, we have :

$$\left[\mu\right] \equiv \left[\frac{1}{v^{2}K}\right] \qquad \textbf{(144)}$$
$$\left[\frac{K}{\mu}\right]^{1/2} \equiv \left[\frac{K}{1/\left(v^{2}K\right)}\right]^{1/2} \equiv \left[K^{2}v^{2}\right]^{1/2} \equiv \left[K\,v\right] \qquad \textbf{(145)}$$

and also because $\left[K\,\mu\right] \equiv \frac{1}{v^{2}}$, we have :

$$\left[K\right] \equiv \frac{1}{\mu\, v^{2}} \qquad \textbf{(146)}$$
$$\left[\frac{K}{\mu}\right]^{1/2} \equiv \left[\frac{1/\left(v^{2}\mu\right)}{\mu}\right]^{1/2} \equiv \left[\frac{1}{v^{2}\mu^{2}}\right]^{1/2} \equiv \left[\frac{1}{v\mu}\right] \qquad \textbf{(147)}$$

So, to sum up :

$$\left[\frac{e}{m}\right] \equiv \left[\frac{K}{\mu}\right]^{1/2} \equiv \left[Kv\right] \equiv \frac{1}{\left[\mu\, v\right]} \qquad \textbf{(148)}$$

Lodge insists that these equations have a universal and objective character: "These relations must hold in any coherent system of units, since they express physical truths."[75] But they

[75] "These relations must hold in any consistent system of units, since they express physical truths;." Lodge 1882: 363.

are clearly not independent of each other. Lodge reminds us that their mutual dependence stems from the three fundamental laws on which his entire discussion is based (Coulomb's electrostatic law, its magnetic counterpart, and Oersted's principle). When this is taken into account, we are left with just two independent relationships, on the one hand :

$$\left[\mu e^2\right] \equiv \left[K m^2\right] \equiv \mathrm{M\,L} \qquad \textbf{(149)}$$

which results from , $\left[e K^{-1/2}\right] \equiv \left[m \mu^{-1/2}\right] \equiv \mathrm{L}\left[F^{1/2}\right]$[76] and , $\left[m\right] \equiv \mathrm{L}\left[\mu e\right]\mathrm{T}^{-1}$[77] and on the other hand :

$$\left[\mu K v^2\right] \equiv 1. \qquad \textbf{(150)}$$[78]

Clearly, the two formulas (64) and (65) summarize, according to Lodge, what can be objectively asserted about the dimensional relationships between the electromagnetic quantities *e*, *m*, *K* and μ - assuming his experiments give the expected result.

Lodge proposes to extend the results obtained above to the case of gravitation, by posing :

$$\left[e K^{-1/2}\right] \equiv \left[m \mu^{-1/2}\right] \equiv \mathrm{L}\left[F^{1/2}\right] \equiv \mathrm{M}\left[k^{-1/2}\right] \qquad \textbf{(151)}$$

where M is a mass, and *k* is a constant that can vary with the medium in which the masses are found, an idea that had already been proposed by Faraday as a consequence of his general idea of interaction by polarization of an intermediate medium. Although Lodge refrains from making any predictions, he does write :

> I'm sure that a direct experimental study of this question would not be without interest, and could perhaps lead to important results.[79]

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[76] See (49).
[77] See (52).
[78] See (63).
[79] "I feel sure that a direct experimental attack on this question would not be uninteresting, and it might lead to important results." Lodge 1882: 359.

This is probably the first time that dimensional considerations have suggested the search for a new effect.

Lodge also asks why Clausius ignores the effect of the medium in the equations. His answer is that the concept of magnetic permeability is not involved in the Amperian theory of magnetism:

> The coefficient μ is therefore foreign to Ampère's theory in its universal version; and this is how Professor Clausius failed to recognize its existence and was misled. A system dealing with Ampère magnets in a medium for which μ is not equal to 1 is an illegitimate combination that is undoubtedly practical on occasion, but can by no means be entirely satisfactory.[80]

The aforementioned difference between continental and British scientific cultures is therefore linked to a difference in perception of Ampère's theory. Lodge defends the British point of view, but contrary to what the above quotation might suggest, he does not consider the inclusion of μ to be incompatible with an Amperian model of magnetism. What he seems to deem an "illegitimate combination" is a model in which microscopic currents would be immersed in a medium itself characterized by a physical quantity μ. On the other hand, he considers Ampère's model appropriate as a microscopic explanation of permeability as a macroscopic property of matter:

> The effect of the medium is a physical fact, and no theory can really claim to dispense with the constant μ. All Ampère's theory does is to give it a physical interpretation, and allow us to dispense with it as soon as we take into account every circuit in the field, molecular or otherwise, and don't arbitrarily limit ourselves to those crude solenoids we can put into action and control directly by batteries.[81]

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[80] "The coefficient μ is thus foreign to Ampère's theory applied universally; and this is how it has happened that Prof. Clausius has failed to recognize its existence and has been led into error. A system dealing with Amperian magnets in media for which μ does not equal 1 is a mongrel combination which may no doubt be occasionally convenient but which never can be thoroughly satisfactory." Lodge 1882: 363.

[81] "The effect of the medium is a physical fact; and no theory can presume really to dispense with the constant μ. All that the Ampèrian theory does is to give a physical interpretation to it, and to render one independent of it so soon as one takes account of every current-conveying circuit, whether molecular or other, existing in the field, and does not arbitrarily elect to deal only with those gross solenoids which we can excite and immediately control by batteries." Lodge 1882: 365.

In modern terms, we can say that he sees permeability as an emergent property of amperian currents. In this, he is in complete agreement with Maxwell, whom he quotes at length to support his own view:

> According to Ampère's hypothesis...the properties of what we call magnetized matter are due to electric currents, so that it is only when we consider a large quantity of substance that our theory of magnetization has course; and if our mathematical methods are able to account for what happens inside individual molecules, they will discover only electric circuits, and we will find magnetic force and magnetic induction always equivalent. However,...in order to be able to make as much use as we like of the electrostatic and electromagnetic systems of units of measurement, we shall retain the coefficient μ, knowing that its value is unity in the electromagnetic system.[82]

Lodge's conclusions seem to have been widely accepted by scientists interested in the controversy over the dimensions of a magnetic pole in the electrostatic system: indeed, the debate ended with his paper. Only one other publication is to be found afterwards, and it is largely in agreement with Lodge's work. This is the paper by Sargant,[83] , whose intention to implement the experiments suggested by Lodge we have already mentioned. Sargant proposes variants of these experiments and states that Lodge's work seems to him to reconcile the views of Clausius and the proponents of Maxwell's dimension formula.[84] He also claims to have already carried out an experiment confirming Lodge's opinion - at the Cavendish laboratory, in June of the same year - but on which he gives no details.[85]

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[82] "According to Ampère's hypothesis...the properties of what we call magnetized matter are due to electric currents, so that it is only when we regard the substance in large masses that our theory of magnetization is applicable; and if our mathematical methods are supposed capable of taking account of what goes on within the individual molecules they will discover nothing but electric circuits, and we shall find the magnetic force and the magnetic induction everywhere identical. In order, however..., to be able to make use of the electrostatic or of the electromagnetic system of measurement at pleasure we shall retain the coefficient μ, remembering that its value is unity in the electromagnetic system." quoted by Lodge 1882: 365.
[83] Sargant 1882: 395-396.
[84] Sargant 1882: 395.
[85] Sargant 1882: 396.

Broadly speaking, then, there are two types of reaction to the conflict between Clausius' and Maxwell's results, with partial overlap. The first is that there is a hierarchy between the equations, that some are more fundamental than others. Larmor and Wead consider the equation that expresses the equivalence between a current and a magnet, and which Clausius takes as the basis of his reasoning, to be secondary to others. While Larmor sees in his hierarchy of equations something objective, for Wead this judgment is purely subjective, and taking Clausius' relation as a starting point is not wrong.

The second approach is to introduce a constant into the relationship employed by Clausius. This is the approach taken by Everett, J.J. Thomson, Lodge, Sargant and even Larmor. According to these authors, Clausius erred in neglecting a dimensional factor, and the electrostatic dimensions of a pole given by Maxwell are the only correct ones.[86]

This second approach put an end to the controversy when Lodge explained it in terms of a cultural difference between the continent and Great Britain: while the former started from Ampère and dispensed with dimensional factors, the latter saw such a factor as an expression of the environment introduced by Faraday. Lodge set about establishing the validity of the British position by proposing experiments likely to confirm the implications of Maxwell's theory concerning this factor. Thus, the problem of inconsistency in dimensional formulas was finally resolved by experimental arguments.

It's worth noting, however, that Lodge is preaching to the converted: it's the British, Maxwellian point of view that he's defending, and the controversy takes place in Great

[86] Instead of reporting Maxwell's reasoning in his treatise, these authors attempt to reconstruct it in various ways. This is certainly due to the fact that Maxwell does not explain how he obtains the formulas for the dimensions of electromagnetic quantities as a function of M, L, T, e and M, L , T, m, from the fifteen dimensional equations he takes as his starting point. See Chapter 3, 114-115.

Britain. It's doubtful whether his arguments would have been as convincing to a German or French audience at the time. French treatises on the subject of dimensions rarely go so far as to indicate the formula for the dimension of a pole in the electrostatic system. When Mascart and Joubert do so in their 1882 work, they side with Maxwell, but this is due to the influence of the latter's treatise rather than to the French taking a stand in the British debate. In fact, the polemic seems to have had no echo in France. Even in Britain, while Lodge's arguments seem to have convinced the protagonists of the debate, his remarks find little echo in treatises aimed at a wider audience of physicists. These treatises simply don't give a dimensional formula for the magnetic pole in the electrostatic system.[87] The electromagnetic dimension formula for this quantity is often given as the only one available, and one author even places it in his section on the electrostatic system.[88] This silence seems attributable to the brevity of the discussions, which are satisfied with electromagnetic-only formulas for magnetic quantities, rather than to a desire to avoid a thorny subject. One exception is Andrew Gray, who in his 1889 treatise retained a symbol for the dimension of permeability in the formula for the dimension of the magnetic pole: $M^{1/2}L^{3/2}T^{-1}\mu^{1/2}$ no doubt because he had read Lodge.[89] In a sense, then, the controversy over the electrostatic dimensions of the magnetic pole will have no repercussions even in Great Britain beyond what we have just discussed. However, it illustrates the difficulties to which the existence of the two systems could lead, and above all it reveals the impact that deep and largely unconscious conceptual disagreements between different scientific cultures could have on dimensional analysis.

[87] Ferguson 1882: 171-173; Sprague 1884: 231; Daniell 1885: 634-635.
[88] Cumming 1876: 222.
[89] Gray 1889: 356.

Chapter 6: Attempt to break with Fourier's tradition

6. 1 - Mercadier and Vaschy's writings

In most of the work we have discussed so far, the notion that the same physical quantity could have different dimensional formulas in different systems of units - notably, the electrostatic and electromagnetic systems - did not shock the authors. This is probably because dimensional considerations developed in the context of conversion between systems of units. Since it was obvious that units varied from one system to another, it seemed natural that dimensions should too.

Early in 1883, this mindset was suddenly challenged by an article by two French telegraph engineers, Ernest Mercadier (1839-1906) and Aimé Vaschy (1857-1899)[90] published in *Annales télégraphiques*;[91] and also, a few months later, in the *Journal de Physique*, which enjoyed greater authority among physicists.[92] Their subsequent publications either summarize this first article or deal with specific aspects.

[90] The author of the famous Vaschy theorem, discussed in Chapter 9.
[91] Mercadier & Vaschy 1883b: 5-16.
[92] Mercadier & Vaschy 1883d: 245-253. The title has been changed, but the article is the same, apart from a few editorial details.

Like a certain elite of French engineers, Mercadier and Vaschy were no strangers to fundamental theories. They were familiar with Maxwell's difficult work. The group of telegraph engineers to which Mercadier and Vaschy belonged was largely responsible for disseminating Maxwell's work in France, especially as it was close to the physics community. Mercadier himself was one of the founding members of the Société Française de Physique. Both he and Vaschy were associated with the Ecole Supérieure de Télégraphie (founded a few years before the publications in question, in 1878), where many elements of Maxwell's Traité were taught. Mercadier was professor of physics there, as well as inspector of studies and director of the laboratory. Vaschy replaced Jules Raynaud, who actively popularized Maxwell's theory in his own publications, and was responsible for translations of British works, including Maxwell's Treatise (translated between 1884 and 1889 by Seligmann-Lui, himself a telegraph engineer). Vaschy drew inspiration from Maxwell's Treatise in his own courses. Mercadier's and Vaschy's ideas are therefore largely influenced by Maxwell. In particular, they accorded the medium (ether and matter) an essential role in the propagation of electromagnetic actions. [93]

In the works that interest us here, Mercadier and Vaschy announce the aim of their approach from the outset in almost provocative terms:

> The various systems of electrical and magnetic units that have been put forward so far, including electrostatic, electrodynamic and electromagnetic systems, are generally referred to as absolute systems, and it is accepted that any of these mutually incompatible systems can be chosen indiscriminately. We propose in this work: 1°to demonstrate that there can rationally be only one system of dimensions for electrical quantities, which cannot be arbitrary, and must be adopted to the exclusion of all others; 2(o) to undertake the search for this system. [94]

[93] Atten 1992, ch.8 in particular 126-131.
[94] Mercadier & Vaschy 1883b: 5.

We can already see that for Mercadier and Vaschy, the concept of dimensions is paramount, rather than that of units: they associate the notion of system with that of dimensions. According to them, the same quantity cannot have different dimensions in different systems, because the dimensions of a quantity reflect its very nature:

> The dimensions of a quantity result...from the nature of that quantity and are its analytical representation. [95]

or :

> The dimensions of the quantity of electricity are the very analytical definition of this quantity: they characterize the physical nature of this quantity as we understand it in the current state of science.
> These dimensions are therefore absolutely *determined* and *unique*.[96]

and again:

> If, moreover, as a result of progress in science, the quantity in question were recognized to be of a different nature from that which had been attributed to it initially, we would, it is true, be obliged to change its dimensions; but it would then be a *different* quantity from the previous one, and we would *have to* give it a *different name*. The other would remain what it was, with its original definition and dimensions.[97]

Mercadier and Vaschy are aware that their readers are deeply accustomed to the opposite view. To overcome this difficulty, the two engineers set out to convince them that they were already unknowingly sharing their conception in another context, that of mechanics:

> When it comes to geometrical and mechanical quantities, we agree, if not on the choice of the three fundamental units, at least on the relationships that exist between the dimensions of these various quantities. If, for example, we take as fundamental units those of length, mass and time, the dimensions of the other quantities are entirely determined; those of a surface are L^2, those of a volume L^3, those of a velocity LT^{-1}, those of an acceleration LT^{-2}, those of a force LMT^{-2}, etc. We don't even think of saying that the fundamental units are L(1) , L($^{2) \text{ or } L(3).}$We don't even think of saying that these are the dimensions of these quantities *in this or that system*; these are their *dimensions* in an absolute way... But if this is true and accepted by all for the mechanical and physical quantities with which we are most

[95] Mercadier & Vaschy 1883b: 6.
[96] Italics by Mercadier. Mercadier 1883c: 465.
[97] Italics by Mercadier & Vaschy. Mercadier & Vaschy 1883b: 6. See also Mercadier 1883a: 71.

> familiar, it must certainly be true for all physical quantities, electrical or otherwise, even those whose nature is less familiar to us; i.e., as soon as any physical quantity is sufficiently defined, either mathematically or in any other way, its dimensions are necessarily determined. Indeed, it is obvious that when such a quantity becomes better known, and its mechanism more familiar to us, its nature will not be changed for that reason, and its dimensions will be as well determined as those of force, energy, etc., etc., etc.[98]

Psychologically, comparison with the dimensions of familiar mechanical quantities is a strong argument: for example, it's hard to imagine speed having any dimensions other than LT^{-1}. The situation in electricity and magnetism seems to stem from confusion over the nature of the quantities concerned, due to their exotic nature, and that of the phenomena considered.

Mercadier and Vaschy, who clearly intend to assume the role of founders, define a series of "principles." The first corresponds to what we've just described, which they describe as a "true axiom":

> The same mechanical or physical quantity, well defined mathematically or experimentally, can have only one dimensional formula, i.e. it can be represented in only one way in terms of the fundamental mechanical units (length, time and mass).[99]

According to Vaschy, at least, the reverse is also true: the dimensions of a physical quantity, being the very expression of its nature, are characteristic of that quantity. Indeed, he states elsewhere:

> Generally speaking, specific dimensions can only relate to a specific physical quantity and are sufficient to determine its nature, if not to explain its inner mechanism. [100]

[98] Mercadier & Vaschy 1883b: 5-6.
[99] Mercadier & Vaschy 1883b: 7.
[100] Vaschy 1883a: 47.

Mercadier and Vaschy's thinking differs profoundly from that of the scientists we discussed earlier, in that the latter all started from *equations* in their reasoning. For them, physical phenomena were conceived above all as *relationships*, of which equations were the analytical representation. For the two engineers, on the other hand, it's the physical *quantities* that come first - and consequently, it's the analytical representation of the latter that they're primarily concerned with. In the event of inconsistency in the dimensions of a quantity, they need to resolve the ambiguity of possible definitions:

> It's true that, at the beginning of the theory, we may hesitate about the definitions to be given to quantities; but it's still necessary to establish these definitions in a complete and invariable way.[101]

At the same time, they repudiate the confusion between units and dimensions:

> Once these definitions have been given, we can choose any units we like, and there's nothing wrong with that from a theoretical point of view; but the dimensions are fixed from then on, and it's a question of deriving them from these definitions. A priori, we may even be led to dimensions other than those of the electrostatic and electromagnetic systems.[102]

Mercadier and Vaschy attribute the inconsistency of the results obtained from equations to the fact that they are not sufficient to determine the dimensions of electrical and magnetic quantities: we have more unknowns than equations, due to the fact that the latter contain coefficients whose dimensions are unknown and "which we can...consider at will as purely numerical coefficients whose value can be arbitrarily fixed, or as quantities endowed with dimensions and having a determined physical meaning."[103]

These latter considerations are nothing new. As we have seen, Maxwell explains in his Treatise that the fifteen dimensional equations he arrives at are not sufficient to

[101] Mercadier & Vaschy 1883b: 7.
[102] Mercadier & Vaschy 1883b: 7.
[103] Mercadier & Vaschy 1883b: 8.

uniquely determine the dimensions of electrical and magnetic quantities as a function of M, L and T.[104] What's more, the use of dimensional constants in the formulas is at the heart of the debate over the dimensions of the magnetic pole.

Mercadier and Vaschy refer in particular to Coulomb's and Ampère's laws, which they express as :

$$f = k\frac{qq'}{r^2} \quad \textbf{(152 a)}$$

and

$$f' = k'\frac{ii'dsds'}{r'^2}(\cos\theta - \frac{3}{2}\cos\alpha\ \cos\alpha')\ . \quad \textbf{(152 b)}$$[105]

They point out that the difference between the electrostatic and electromagnetic systems is that in the former, k is taken to be equal to 1, while in the latter, k' is.[106] It will be remembered that in 1863, the Committee of the *British Association* had in fact defined the electromagnetic system on the basis of the equation giving the force between an electric current and a magnetic pole, not between two currents.[107] Mercadier and Vaschy chose to use the latter (which forms the basis of Weber's "electrodynamic system") because they considered the equivalence between current and magnet to be universally shared. Vaschy writes elsewhere:

> According to the ideas universally accepted today, the theory of electro-magnetic or magnetic attraction and induction would be entirely analogous to that of electro-dynamic phenomena. A solenoid is often regarded as a magnet, and vice versa. It follows that, since the coefficient k' of Ampère's formula is independent of the nature of the media, the analogous coefficients of the electro-magnetic and magnetic formulas would also be independent of it.[108]

[104] See chapter 3, 114-115.
[105] Mercadier & Vaschy 1883b: 8.
[106] Remember that Maxwell and Jenkin defined the electromagnetic system in terms of the force between a current and a magnet, not between two currents. Nevertheless, since a current is considered equivalent to a magnet, it amounts to the same thing.
[107] See chapter 3.
[108] Vaschy 1883a: 117.

But the real originality of the two engineers lies in their rejection of the approach of arbitrarily assigning the value 1 to coefficients:

> This optional deletion of the coefficient in one of Coulomb's or Ampère's formulas, while not inconvenient from a practical point of view, is inadmissible from a theoretical point of view. These formulas, together with the definition $q = it$, show that the ratio $\frac{k}{k'}$ independently of any hypothesis, is endowed with dimensions and equal to the square of a velocity . L^2T^{-2} [109] This ratio is therefore a physical quantity with a specific meaning.... It follows that at least one of the coefficients k and k' is something other than a purely numerical coefficient; and a priori they must both be treated as physical quantities with an absolute and non-optional meaning.[110]

These considerations lead Mercadier and Vaschy to formulate what they call the second principle:

> From a theoretical point of view, we have no right to suppress a priori the coefficients of Coulomb's and Ampère's formulas and all those deduced from them, without first attempting to determine their nature.[111]

These concerns are reminiscent of those of Lodge, as discussed in the previous chapter, as well as those of Maxwell and Jenkin when, in 1863, they defined systems taking into account the influence of the environment.[112]

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[109] Mercadier and Vaschy raise this to the level of a theorem: "We can then state the following general theorem: If we accept the definition of intensity $q = it$, and if electrostatic and electromagnetic forces are of the same nature as the forces considered in Mechanics and characterized by the dimensions $LMT^{(-)(2)}$, the ratio of the coefficients which enter into Coulomb's and Ampère's laws necessarily represents, to within one numerical factor, the square of a velocity." (Mercadier & Vaschy 1883a: 120).

[110] Mercadier & Vaschy 1883b: 8-9. See also Mercadier & Vaschy 1883a, 118.

[111] Mercadier & Vaschy 1883b: 9.

[112] Elsewhere, Vaschy treats the gravitational constant as a dimensionless number: "If, instead of calling *force* the product of a mass and the acceleration it undergoes, we had, on the basis of Newton's law $f = k\frac{mm'}{r^2}$, given this name to the quotient of the square of a mass and the square of a length, - instead of dimensions: $[F] = LMT^{-2}$ we would have $[F] = L^{-2}M^2$." Vaschy explains that this force would be a different physical quantity, since its dimensions are different; however, for him it goes without saying that k has no dimensions. That he considers the gravitational force to be different in nature from a force defined by Newton's second law is certainly surprising, given that elsewhere he attributes dimensions LMT^{-2} to electric and magnetic forces. This is clearly because he thinks that these coefficients are of a different nature from k, no doubt

Indeed, Mercadier and Vaschy believe that the question of coefficient variation can only be answered by experiment. At the end of their article in *Annales Télégraphiques*, they suggest such experiments, which they intend to carry out:

> To find these dimensions [those of electrical and magnetic quantities], it is necessary to resort to experiment instead of arbitrarily treating the coefficients of Coulomb's and Ampère's formulas, which have an absolute meaning and not an optional one as is usually admitted.[113]

And like Lodge, they believe that the whole point is to determine the role of the medium, represented by the coefficients: "*k* and *k*' are the coefficients of intervention of the medium in the two formulas."[114] Whether they were actually aware of Lodge's work is not certain, but in any case, more fundamentally, their insistence on the medium is clearly inspired by Maxwell's work.

The two engineers therefore propose to determine the dimensions of *k* and *k'* experimentally. As far as *k is* concerned, they note that it is the inverse of the specific inductive power of the medium. It therefore varies with the medium, making the electrostatic system untenable in their eyes. To determine the dimensions of *k*, they refer to suggestive but not definitive experiments that Ludwig Boltzmann and others had carried out to verify Maxwell's relationship between optical index and dielectric permittivity:

> In studying the specific inductive powers of various transparent dielectric media, various physicists have observed that these powers are proportional to the square of the refractive indices of light. *If* this law *were* true, we *would conclude* that the inverses, i.e. the values of *k*, are proportional to the square of the speed *V* of light in these media. The ratio $\frac{k}{V^2}$ *would* then *be* an absolute constant, independent not only of the quantities of electricity present and their relative positions, but also of the nature of the medium. Or: $\frac{k}{V^2} = \alpha$. Coulomb's formula could then be written :

because he sees in their dimensions the effect of the medium, unlike in the gravitational case. Vaschy 1883a: 47.

113 Mercadier & Vaschy 1883b: 12.

114 Mercadier & Vaschy 1883b: 9.

$$f = \alpha V^2 . \frac{qq'}{r^2},$$

α being really a numerical coefficient devoid of physical meaning, which we could take equal to 1 to define the unit of electricity quantity.[115]

Mercadier and Vaschy deduce the dimensions of the charge, $L^{1/2}M^{1/2}$, and conclude that in this case, the electromagnetic system would then be "the rational system that alone must exist." We can see that for both engineers, if a constant is independent of the medium, this necessarily implies that it is dimensionless.[116]

Mercadier and Vaschy give no further details in their article in *Annales Télégraphiques*.[117] They simply insist that their conclusion depends on two conditions: firstly, a clear definition of the refractive index used to make the comparisons, which is not self-evident given that the refractive index varies with the wavelength of light; secondly, that the law in question is indeed verified. Mercadier and Vaschy believe this to be probable, and intend to verify it for themselves. They developed their reasoning in a publication in the *Comptes Rendus* de l'Académie des Sciences, pointing out that the capacitance of a capacitor is inversely proportional to the constant *k* (inverse of permittivity). [118]

Mercadier and Vaschy then turn to the question of the dimensions of the *k'* coefficient. In their note to the *Comptes rendus*, they follow up their derivation of the relationship $k = \alpha V^2$ with the remark:

If this is the case, in the ratio $\frac{k}{k'}$, which represents the square of a speed to within one numerical factor, *k'* would be a number, independent of the fundamental units.

[115] Mercadier & Vaschy 1883b: 10-11, italics by Mercadier and Vaschy.
[116] Mercadier & Vaschy 1883b: 11.
[117] Mercadier & Vaschy 1883b: 5-16.
[118] Mercadier & Vaschy 1883a: 120.

> From then on, we would be entitled to use this number equal to unity in Ampère's formula to define the unit of current, and the use of the electromagnetic system of electrical units would be justified theoretically, as it already is by practical considerations.[119]

This line of reasoning seems to imply that k' is a *constant*, independent of the medium. Mercadier and Vaschy have already carried out a series of experiments showing that k' remains the same for bodies whose specific inductive power varies considerably. Because of the limited precision of their measurements, they are reluctant to draw any general conclusions. In the event of further confirmation, they conclude that k' "must...be regarded as a purely numerical coefficient devoid of physical meaning"[120] and that the electromagnetic system is the only correct one. Note that this is in line with the Amperian conception of magnetism that Mercadier and Vaschy share with Maxwell: given that the magnetic properties of the medium on the macroscopic scale are ultimately due to molecular currents on the microscopic level, it follows that the only real fundamental property of the medium is its dielectric permittivity. Mercadier and Vaschy therefore expect k' to be constant.

The series of experiments referred to here was carried out in January 1883 at the Ecole Supérieure de Télégraphie. It involves the phenomenon of induction.[121] For Mercadier and Vaschy, determining experimentally whether k' is a physical quantity with dimensions, or whether it can be taken to be equal to 1, means determining whether electrodynamic forces depend on properties of the medium other than its permeability.

[119] Mercadier & Vaschy 1883a: 120-121.
[120] Mercadier & Vaschy 1883b: 12.
[121] Mercadier & Vaschy 1883b: 12-16. This series of experiments is also described in Mercadier & Vaschy 1883b: 250-252, and in Vaschy 1883a: 116-118. In addition, Mercadier and Vaschy express their intention to carry out further series of experiments to verify their results.

Vaschy remarks elsewhere: "When we say that magnetic actions are independent of media, we must make a restriction and except magnetic media."[122] Mercadier and Vaschy therefore set out to determine the influence of media with very low permeability - ideally, it should be zero, but this is impossible in practice.

The two engineers have built an induction coil consisting of two circuits separated by an empty space, which can be filled with various fluids when the coil is immersed in them. The primary circuit is connected to a battery, and the secondary to a ballistic galvanometer. Mercadier and Vaschy start from the relationship :

$$Q = \frac{Mi}{R} + \frac{\sum \left(M_{\mathrm{n}} I_{\mathrm{n}}\right)_o^\theta}{R}, \tag{153}$$

where Q is the quantity of electricity induced (implied during the establishment of the primary current)

M the mutual induction coefficient between the two circuits

i the final current in the field circuit (primary)

R *armature* resistance

I_n: current intensity in a neighboring circuit

M_n: induction coefficients of another neighboring circuit on the induced circuit

$\sum \left(M_{\mathrm{n}} I_{\mathrm{n}}\right)_o^\theta$ variation of $\sum \left(M_{\mathrm{n}} I_{\mathrm{n}}\right)$ over the duration θ of the induction phenomenon. of induction.[123]

In this expression, we can see that the first term represents the induction between the two circuits of the coil, and also that Mercadier and Vaschy added a second term to take into account the effect of any circuit - or, equivalently, magnet - in the vicinity of the coil.

They consider this formula incomplete, as it does not take into account the coefficient *k'*. To introduce the latter, they reason from Ampère's formula:

$$f' = k' \frac{ii' \, ds ds'}{r'^2} (\cos\theta - \frac{3}{2} \cos\alpha \ \cos\alpha'). \tag{154}$$

[122] Vaschy 1883a: 118.

[123] Mercadier & Vaschy 1883b: 13-14.

Introducing k' into Ampère's law therefore means replacing i and i' by $i\sqrt{k'}$ and $i'\sqrt{k'}$ respectively. By analogy, Mercadier and Vaschy introduce it into formula (i) by replacing i and i' respectively by $i\sqrt{k'}$ and $i'\sqrt{k'}$, and obtain :

$$Q = \left(\frac{Mi}{R} + \frac{\sum (M_n I_n)_o^\theta}{R} \right) \sqrt{k'} . \qquad \textbf{(155)}$$

They simply measure Q when the coil is immersed in different media: air, alcohol, oil, petroleum and glycerine. They describe three series of experiments, with coils of different numbers of turns and using batteries of different voltages.

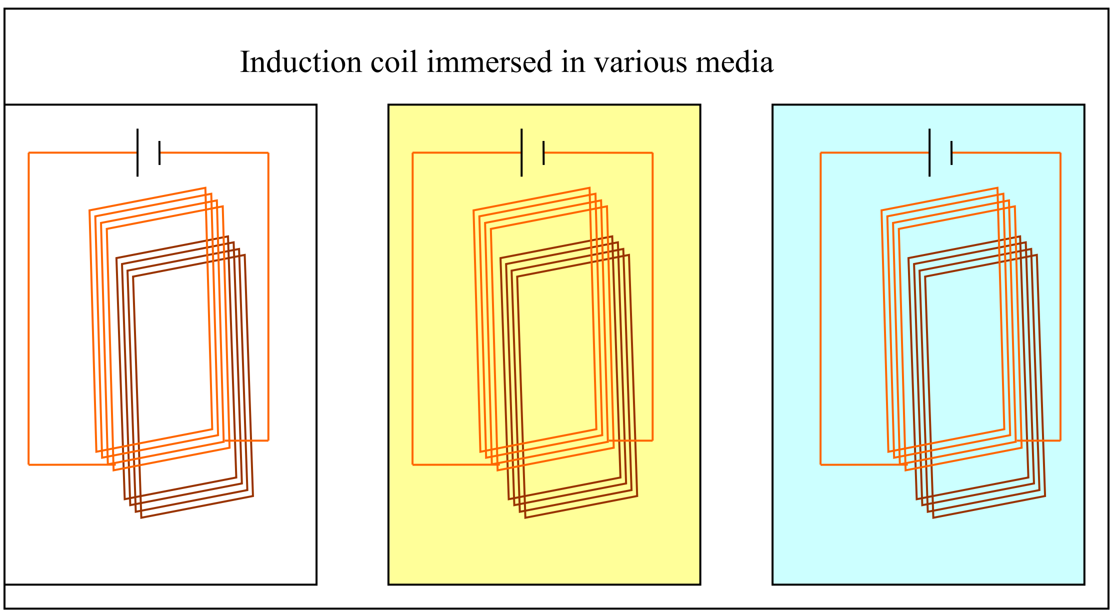

Figure 6.1

In two of the three series of experiments, they observe no difference in the value of Q depending on the medium, and in the third, the difference they observe is so small that they choose to neglect it (it is about 1/300). [124]

[124] These results are due to the fact that their experiments are not precise enough to detect variations due to permeability, which in reality depends on the medium.

In another publication, the two engineers suggest further experiments to confirm that the characteristic coefficient of magnetic phenomena is indeed an absolute constant: [125]

- In the first, a magnetized needle is placed under a glass cage (to protect it from air currents), and a magnet is placed nearby. When non-magnetic masses are moved close to this arrangement, no influence is observed on the needle, in particular no change in the action of the magnet on it.

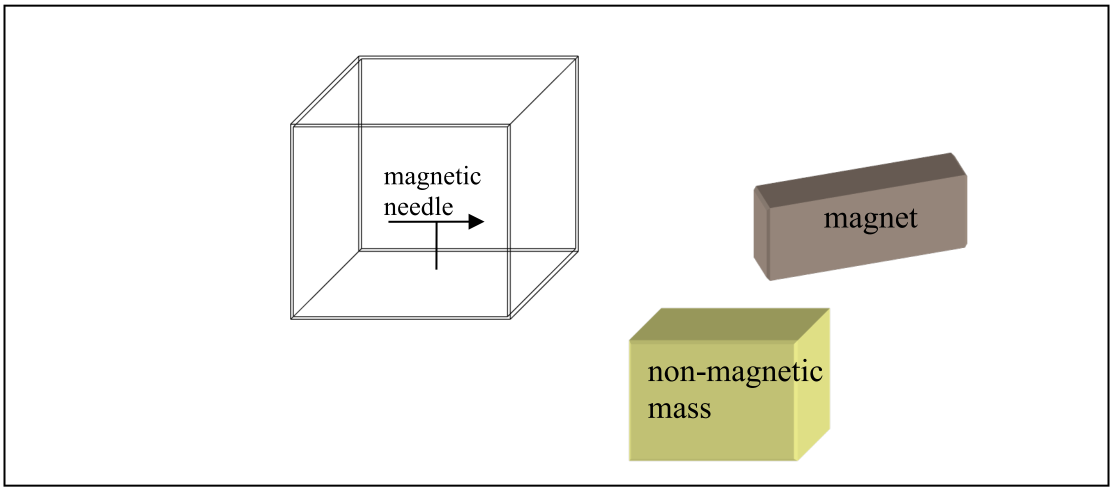

Figure 6.2

- The second experiment involves a magnet contained in a glass envelope filled alternately with water and oil. A galvanometer placed near the magnet indicates the same value in both cases.[126]

[125] Mercadier & Vaschy 1883b: 252-253.
[126] Mercadier & Vaschy 1883b: 252-253.

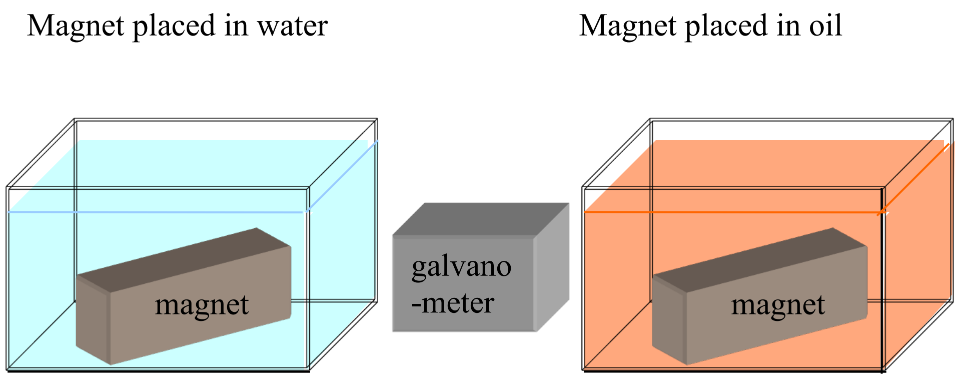

Figure 6.3

- In the last experiment, we place not only a magnet, but also a magnetized needle in turn in petroleum and glycerine. The needle's deflection is measured, and found to be the same in both cases. [127]

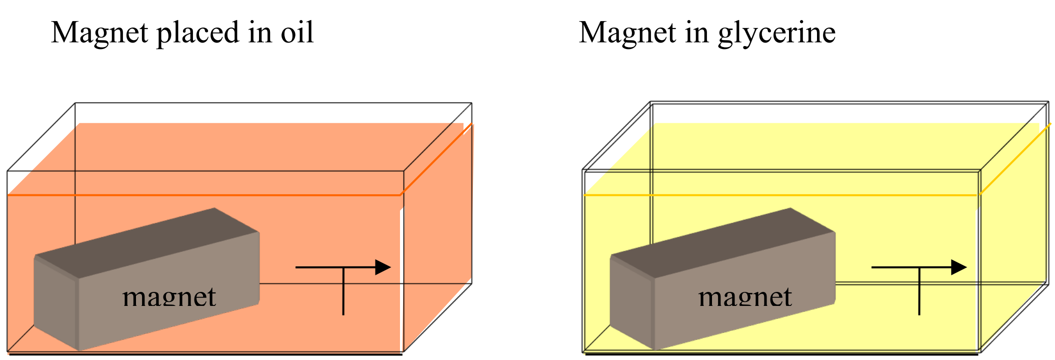

Figure 6.4

The first of these experiments may come as a surprise, but Vaschy explains the reason elsewhere:

[127] Mercadier & Vaschy 1883b: 252-253; Vaschy also describes this last experiment in another publication: Vaschy 1883a.

> The demonstration follows from the fact that the reciprocal actions of magnetic masses are only modified by the approach of magnets or particular substances known as magnetics.[128]

As for the other two experiments, it goes without saying that their purpose is to test the possible influence of the environment on the magnetic force.

These experiments are somewhat reminiscent of those proposed by Lodge. The latter imagined exploring whether or not a circuit and a magnet producing the same magnetic field in air retained this equivalence between them in different media (and in the second case, how the field differed). Mercadier and Vaschy, on the other hand, were simply interested in the variation of a magnet's field in different media, without focusing on the question of the equivalent current field. What's more, Lodge considered magnetic media, whereas Mercadier and Vaschy assumed pure dielectrics.

Based on the results of the experiments described above, the two engineers draw the following conclusions, this time with relative confidence:

> It seems safe to assume: 1°that the coefficient *k*' of the electrodynamic formulas is an absolute constant; 2°that the same is probably true of the coefficient of the electromagnetic formulas; 3°that since *k*' is an absolute constant, the coefficient *k* of Coulomb's law is the square of a velocity.[129]

More generally, it seems that Mercadier and Vaschy tend to confuse the notion of a dimensionless quantity with that of a "constant." On the one hand, they believe that if a quantity like k' has no dimension, it must be a constant. On the other hand, they believe that if a quantity like k' has no dimension, it must be a constant. Perhaps this is due to a conception of experiment as the ultimate jury of any assertion in physics, or more likely to

[128] Vaschy 1883a: 118.
[129] Mercadier & Vaschy 1883b: 16.

the fact that the notion that k' is dimensionless depends on the constancy of $\frac{k}{V^2}$, which Boltzmann's experiments only approximately established.[130]

The same identification between absolute constant and dimensionless parameter can be found in Mercadier and Vaschy's analysis of their experiments. This time, they reason in the opposite direction: a constant cannot have dimensions. Indeed, they state that if k' turns out to have the same value whatever the medium considered, then it would be an "absolute constant which must consequently be considered as a purely numerical coefficient devoid of physical meaning".[131]

This preconception that an absolute constant has no dimensions may seem surprising. In fact, nowadays most universal constants are given dimensions. To give just one example, the gravitational constant has dimensions $\mathrm{L}^3\mathrm{M}^{-1}\mathrm{T}^{-2}$. Vaschy, on the other hand, takes it for granted that it has no dimensions when he says that if we had chosen to call the quantity defined by Newton's law of gravitation a "force", its dimensions would be : [132]

$$[F] = \mathrm{L}^{-2}\mathrm{M}^2 . \quad \textbf{(156)}$$

Furthermore, as mentioned above, Mercadier and Vaschy state that the two coefficients k and k' were considered "at will as purely numerical coefficients whose value can be arbitrarily fixed, or as quantities endowed with dimensions and having a definite physical meaning."[133] This statement shows the extent to which they establish a close link between the dimensions of a quantity and its physical nature, to the extent that when a quantity - if

[130] Buchwald 1994: 212-213.
[131] Mercadier & Vaschy 1883b: 12.
[132] Vaschy 1883a: 47.
[133] Mercadier & Vaschy 1883b: 8.

it can still be called such - has no dimensions, it has no physical meaning. This may seem an extreme view. For example, can we say that the atomic number of an element and the azimuth of a planet have no physical meaning?

In contrast to Vaschy and Mercadier's point of view, we have seen that Stoney advocates taking the gravitational constant as one of the fundamental quantities of his system.[134] Without going that far, it's doubtful that this constant has no physical significance.

Finally, we may wonder to what extent Mercadier and Vaschy's identification of a numerical coefficient with a medium-independent constant was shared at the time. British scientists who used[135] coefficients attributed dimensions to a coefficient in one system, and regarded it as numerical in the other. As the absence of dimensions was not an objective characteristic of a coefficient, it was difficult to see how they could associate it with the variation of the coefficient as a function of the environment.

We can only guess at the reasons behind Mercadier and Vaschy's conflation of the numerical and the constant. We mentioned above that their work represented a break with the position of attributing to the same quantity several formulas of different dimensions in different systems, a position linked to the context of conversions between systems of units. Ironically, their amalgam between constant and numerical is probably linked to the same question of changes of units. When changing units, in mechanics at least,[136] the conversion factor is numerical.[137] It would seem that Mercadier and Vaschy have unconsciously

[134] See chapter 4.
[135] See table 4.1 above, by Everett.
[136] This is not the case for electrical and magnetic quantities when we switch from the units of the electrostatic system to those of the electromagnetic system, as in Maxwell's case. But Mercadier and Vaschy disagreed with Maxwell on this point, demanding that these quantities have the same dimensions in both systems.
[137] Or at least it can be considered as such. It's true, after all, that to convert minutes into seconds, the conversion factor 60 represents "seconds per minute." However, to consider the latter expression as a unit is

deduced that the opposite is true, i.e. that a purely numerical factor is always the arbitrary consequence of the size of the units we have given ourselves, and therefore cannot represent a physical quantity.

6. 2 - The reception of Mercadier and Vaschy's ideas :

Despite their provocative nature, Vaschy and Mercadier's positions did not provoke an outcry. They did, however, meet with some criticism, notably from Maurice Lévy.[138] The latter set out to demonstrate that the two engineers were wrong to conclude that the coefficient k' of Ampère's formula is equal to 1. Lévy points out that their demonstration is based on two ideas. Firstly, dimensional considerations lead to the conclusion that the ratio $\frac{k}{k'}$ represents the square of a velocity, which he denotes by W. We therefore have :

$$\frac{k}{k'} = W^2. \qquad \textbf{(157)}$$

Secondly, Maxwell's experiments suggest that the coefficient k is characteristic of a given medium, and is proportional to the square of the speed of light in that medium (when transparent, of course). If we denote this speed by V, we get

$$k = \alpha V^2, \qquad \textbf{(158)}$$

where α is a constant independent of the medium.

Lévy accepts these steps in Mercadier and Vaschy's reasoning. However, he explains that by then taking k' *to* be equal to 1, they are implicitly making the assumption that W and V are proportional, i.e. that the ratio $\frac{k}{k'}$ in various media is proportional to the

totally artificial: it doesn't refer to a physical quantity, whose numerical value if expressed in this "unit" would be one.

[138] Lévy 1883: 248-250.

square of the speed of light in each of these media. Lévy considers this unlikely, for reasons that have to do with his conception of physical forces. He believes that all forces act through a medium, and therefore considers it unlikely that k' can be independent of the medium. What's more, since he thought that the force involved in Ampère's formula was different in nature from electrostatic forces, he doubted that the phenomena underlying them in the medium were the same:

> If, as indicated by the variability of the *k* coefficient, it is the presence of a continuous medium that gives rise to the forces that appear to us as actions at a distance, this medium must intervene in all actions, whether they are electrostatic or electrodynamic in essence; if it intervenes in both kinds of action by one and the same quality, for example by its density, it would follow that in passing from one medium to another, the *k* and *k*' coefficients should change in the same ratio. But this hypothesis, although less improbable a priori than the one which leads the authors [Mercadier and Vaschy] to regard the medium as not intervening at all in the production of electrodynamic or magnetic actions, is not probable either. The ratio $\frac{k}{k'}$ undoubtedly depends on the medium, but without the velocity whose square it represents being equal or proportional to that shown in formula (1) [$k = \alpha V^2$], one of which is the speed of light, the other possibly being the speed of propagation of vibratory movements of a different nature to which the medium is susceptible, and the ratio of the two having to depend on the magnetic and dielectric polarization coefficients of the medium.[139]

Whatever Lévy's conceptions of the physical phenomena involved, his criticism of Mercadier and Vaschy's reasoning remains pertinent. When he notes that the equation $k' = \alpha \frac{V^2}{W^2}$ implies that k' is a constant independent of the medium only on condition that V and W are proportional, he identifies a real weakness in Mercadier and Vaschy's reasoning. Precisely because they had reasoned purely in terms of dimensions, they were unable to realize that they had assumed the proportionality of two velocities of a different nature.

139 Lévy 1883: 250.

Mercadier and Vaschy replied to Lévy a few months later. As we have seen, it is the experimental determination of whether or not *k*' varies with the medium that they attach the most weight to. This is no doubt why they insist in their reply on the results of the experiments they have just published: [140]

> On examining them, we may not find the results, which are negative in form, sufficient to be able to affirm absolutely that the coefficient *k*' of Ampère's formula is a numerical constant; but we will recognize that they are not such as to make us abandon our predictions.[141]

They don't bother to respond to Lévy's criticisms of their implicit assumption of the proportionality of speeds *V* and *W*, probably because they don't affect the conclusions they draw directly from the measurement of *k*'. Moreover, Lévy agrees with Mercadier and Vaschy on the importance of their experiments: "[The] experiments that the learned authors promise can alone decide, in the last resort, in this matter."[142] Not everyone agrees on this point, however. As we shall soon see, a certain J. Boulanger was skeptical about Mercadier and Vaschy's experiments.

6. 3 - The question of systems of units after Mercadier and Vaschy's intervention

Although it is difficult to estimate the exact impact of Mercadier and Vaschy's series of articles beyond the immediate reactions they generate, it is clear that they do not put an end to the debate on the question of systems of electromagnetic units. In France at least, many agree with them that the multiplicity of systems is unsatisfactory for theoretical

[140] Mercadier & Vaschy 1883c: 334-336.
[141] Mercadier & Vaschy 1883c: 335.
[142] Lévy 1883: 250.

as well as practical reasons, that there is in fact only one legitimate dimensional formula for each coefficient (or physical quantity), that the multiplicity of systems is due to ignorance of these dimensional formulas for the coefficients involved, and that it would be desirable to put an end to it by determining once and for all what these formulas are. In this respect, Mercadier and Vaschy's conceptions seem to find an echo. On the other hand, their concrete proposals for solving the problem are not unanimous.

There are essentially four attitudes. Some propose studies on the foundations of the systems; others (including Mercadier) seek to establish dimensional relations whose validity does not depend on the system under consideration; still others decide to assume ignorance of the dimensions of the coefficients by adding fundamental quantities; and finally, some attempt to determine the dimensions of the coefficients not experimentally, but on the basis of theoretical criteria.

6.3.1 – Studies on the foundations of systems

Two years after Mercadier and Vaschy's publications, in 1885, a certain J. Boulanger set out to examine the foundations of unit systems.[143] His approach consisted in listing the physical quantities involved on the one hand, and the equations expressing the relationships between them on the other. He includes in the latter all coefficients likely to play a role. This approach enables him to analyze the situation in terms of indeterminacy: the problem arises because there are more physical quantities (including coefficients) than equations. Boulanger takes six physical quantities in addition to the four coefficients: quantity of electricity (Q), capacitance (C), electromotive force (E), resistance (R), current intensity

[143] Boulanger 1885: 3-7.

(I), and "magnetic mass" (q). His relations are based on four equations that he considers "definitions" (that of current $Q = I\,t$, that of capacity $Q = C\,E$, Ohm's law and Joule's law) and four others that are "expressions of laws deduced from experience"[144] and contain the coefficients (Coulomb's two formulas, Ampère's formula and Biot-Savart's law). We therefore have ten quantities for eight relations. To break this deadlock, we take into account only two of the four coefficients, taking the other two to be equal to one. Boulanger implicitly excludes the possibility of dimensionless variable coefficients. He considers all possible combinations of coefficients - just eliminating the one that doesn't involve magnetic mass. In all, he has five systems (including the electrostatic and electromagnetic systems), in each of which he determines the dimensions of the six quantities, and discusses their units. Unlike Mercadier and Vaschy, he does not attempt to choose which of the possible systems is actually correct:

> From a theoretical point of view, these systems...are all equally valid, provided that in choosing one of them we take care to specify the conditions under which the actions take place.[145]

For him, the choice of a system is dictated purely by considerations relating to a particular experimental context:

> We are obliged to make do with systems based on certain assumptions, which are therefore only accurate under special conditions. It is therefore necessary to specify these conditions with each system; in a word, we will have the right to equal to unity any two of the coefficients, provided we do not vary the medium in which the actions take place. [146]

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[144] Boulanger 1885: 4.
[145] Boulanger 1885: 7.
[146] Boulanger 1885: 4.

Basically, his views are in line with those of Mercadier and Vaschy: he believes that the choice of a system is impossible in the contemporary state of knowledge, but that it is desirable and possible in principle:

> This indeterminacy would disappear if we knew the intimate mechanism of electrical and magnetic actions, for then the coefficients could be expressed in terms of two new quantities, such as specific inductive power and magnetic power. These quantities, linked to the mechanical properties of the media, would have definite dimensions, and we would then have eight equations to determine eight unknowns. This would be the unique absolute system. [147]

As mentioned above, Boulanger doubted Mercadier and Vaschy's experiments. The reason for this is not entirely clear, but it would seem that, despite his declarations of principle regarding the contemporary ignorance of the phenomena involved, he believes that neither k nor k' can be dimensionless:[148]

> In reality, this system [the electromagnetic system] is no better than the electrostatic system, since the intermediary medium is just as important for its magnetic power as for its specific inductive power, and if the experiments gave the result we have just reported [coefficient independent of medium], this only proves that the media tested had a magnetic power little different from that of air. [149]

6.3.2 - Search for dimensional relationships independent from the system under consideration

In 1893, we find a series of works in which results are established that are intended to be correct regardless of any assumptions as to the exact dimensions of the coefficients under discussion, and therefore independently of the system of units considered. It is unlikely

[147] Boulanger 1885: 4.

[148] Boulanger seems to have misunderstood Mercadier and Vaschy's approach, believing that their experiments concerned the coefficient of Coulomb's formula for magnetic poles. Nevertheless, his opinion would clearly be the same as regards k'.

[149] Boulanger 1885: 4.

that these works were entirely independent of one another. Although we can't say for sure, it would appear that the source was a publication by Mercadier dating from the first half of 1893. [150]

From Coulomb's law, Laplace's law and Ampère's formula, Mercadier derives the following dimensional relationships between the coefficients involved in these laws:

$$\frac{[k]}{[a]} = \mathrm{L}^2\mathrm{T}^{-2} \qquad \frac{[k][k']}{[\lambda]^2} = \mathrm{L}^2\mathrm{T}^{-2} \qquad [\lambda]^2 = [\mathrm{a}][k'] \tag{159}$$

where k, k', λ and a denote respectively the coefficients of Coulomb's laws for electrostatics and magnetism, the coefficient of Laplace's law which gives the force between a pole and a current, and that of Ampère's formula (note that Mercadier uses a different symbol for the latter than in his articles with Vaschy, where this coefficient was denoted by k'). [151] These dimensional relationships do not prejudge the exact dimensional formulas of the coefficients:

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[150] This is a series of three articles in the *Comptes Rendus de l'Académie des Sciences*, also published as one in the *Journal de Physique*. Mercadier 1893a: 800-803; Mercadier 1893c: 872-875; Mercadier 1893d: 974-977; Mercadier 1893b: 289-299.

[151] Mercadier starts from $f = k\frac{q\,q'}{r^2}$ and $f = k'\frac{\mu\,\mu'}{r^2}$ and expresses the quantities of electricity and magnetism, q and μ, in terms of the dimensions of k and k' and L, M and T initially, then in terms of the dimensions of a force, MLT^{-2}. He thus obtains:

$q = n\,\mathrm{K}^{(-1/2)}\mathrm{M}^{(1/2)}\mathrm{L}^{(3/2)}\mathrm{T}^{(-1)} = n\,\mathrm{K}^{-1/2}\mathrm{L}\sqrt{\mathrm{F}}$ and $\mu = n'\,\mathrm{K'}^{(-1/2)}\mathrm{M}^{(1/2)}\mathrm{L}^{(3/2)}\mathrm{T}^{(-1)} = n'\,\mathrm{K'}^{-1/2}\mathrm{L}\sqrt{\mathrm{F}}$

where n and n' are numerical constants. He deduces dimensional relationships between coefficients from other relevant equations: from the definition of intensity $\mathrm{i} = \mathrm{n}\frac{\mathrm{q}}{\mathrm{t}}$ and from Ampère's formula

$f = \mathrm{a}\frac{\mathrm{ii'\,ds\,ds'}}{\mathrm{r}^2}(2\cos\theta - 3\cos\alpha\cos\alpha')$, he gets $\frac{\mathrm{k}}{\mathrm{a}} = \mathrm{n}\,\mathrm{L}^2\mathrm{T}^{-2}$.

From Laplace's law $f = \lambda : \frac{\mu\,\mathrm{i}\,\mathrm{ds}\sin\alpha}{\mathrm{r}^2}$: $\frac{\mathrm{k\,k'}}{\lambda^2} = \mathrm{n'}\,\mathrm{L}^{(2)}\,\mathrm{T}^{(-2)}$. Comparing these two relations and eliminating k, he arrives at : $\lambda^2 = \mathrm{N\,a\,k'}$. From a purely dimensional point of view - i.e. ignoring the numerical coefficients - we thus obtain the dimensional relationships between the coefficients. Only later does Mercadier mention Ohm's law and Joule's law, as well as the phenomenon of induction, to note that they do not introduce any new coefficients. Mercadier 1893a: 800-803.

> The general relationships...that exist between them [the four coefficients] prejudge nothing about their physical nature: whether they are numerical constants or physical quantities expressible in lengths, times and masses, the above relationships are always true; they can be considered as *necessary* consequences of the very definitions of the quantities q, μ, i, and of the mathematical *form of* Coulomb's, Ampère's and Laplace's laws.[152]

Furthermore, Mercadier's result implies that all dimensions of electric and magnetic quantities (including electric charge q and magnetic pole μ) can be expressed using the coefficients k, k', a and λ :

> It follows from the above considerations that each electrical quantity can be expressed in dimensions in several ways as a function of: either k, or a, or λ and k', or k and a, or λ, k and k', etc., or their units.[153]

In the same year, Félix Lucas[154] and Claude Clavenad[155] established dimensional relationships equivalent to those of Mercadier. Although they were inspired by his work, their reasoning differs slightly. Whereas Mercadier looks for the dimensions of q and μ in terms of those of the coefficients, Lucas does the opposite: he expresses the dimensions of

[152] Mercadier 1893a: 803.
[153] Mercadier 1893a: 803.
[154] Lucas 1893: 389-392.
[155] Clavenad 1893: 551-555.

the coefficients in terms of those of the electric charge and the quantity of magnetism.[156] The result he obtains agrees with Mercadier's: [157]

$$k\,\mathrm{L}^{-2}\,\mathrm{T}^{2} = a = \lambda = k'. \qquad \textbf{(160)}$$

Clavenad, for his part, takes the four equations in which the coefficients are involved, but makes use of the fact that their left-hand members all represent a force, so that the right-hand terms are homogeneous among themselves:

$$\underset{\substack{\text{Ampère}\\(1)}}{[a][I]^2} = \underset{\substack{\text{Laplace}\\(2)}}{\frac{[\lambda]\mathrm{M}[I]}{\mathrm{L}}} = \underset{\substack{\text{Coulomb}\\\text{(magnetism)}\\(3)}}{\frac{[k']\mathrm{M}^2}{\mathrm{L}^2}} = \underset{\substack{\text{Coulomb}\\\text{(electrostatics)}\\(4)}}{[k]\frac{[q]^2}{\mathrm{L}^2}}, \qquad \textbf{(161)}$$

where the different symbols represent the dimensions of the corresponding quantities. Lucas and Clavenad obtain few interesting results beyond these relationships. Clavenad goes on to take $[q]$, L and T as fundamental quantities, and attempts to determine the exact dimensions of the coefficients, but with questionable reasoning (which leads him to conclude that a is a constant and k has the square of a velocity as its dimensions).

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156 Lucas represents these quantities respectively as ε and μ. Like Mercadier, he expresses them in terms of the dimensions of a force.

$$\underset{(a)}{[k'] = F\,\mathrm{L}^2\mu^{-2}} \qquad \underset{(b)}{[k] = F\,\mathrm{L}^2\varepsilon^{-2}} \qquad \underset{(c)}{[a] = F\,\mathrm{T}^2\varepsilon^{-2}} \qquad \underset{(d)}{[\lambda] = F\,\mathrm{L}\,\mathrm{T}\,\varepsilon^{-1}\mu^{-1}}$$

In order to obtain relationships between the dimensions of the coefficients themselves, he uses the fact that we know the dimensions of $\frac{\mu}{\varepsilon}$, i.e. $\mathrm{L}^2\mathrm{T}^{-2}$. He therefore takes $\frac{(b)}{(a)}$; $\frac{(c)}{(a)}$; $\left(\frac{(d)}{(a)}\right)^2$ and obtains :

$$\frac{k}{k'} = \frac{a}{k'}\mathrm{L}^2\mathrm{T}^{-2} = \left(\frac{\lambda}{k'}\right)^2 \mathrm{L}^2\mathrm{T}^{-2} = \left(\frac{\mu}{\varepsilon}\right)^2 = \mathrm{L}^2\mathrm{T}^{-2}$$

157 He multiplies by : $k'\,\mathrm{L}^{-2}\mathrm{T}^2$ $k\,\mathrm{L}^{-2}\mathrm{T}^2 = a = \frac{\lambda^2}{k'} = k'$ from which: $\lambda = k'$ he substitutes in the previous expression.

Mercadier, on the other hand, for the first time proposed different systems of dimensions - not units. He himself insists heavily on this difference. At the end of his talk, he declares:

> There was no mention...of systems of electrical *units*. Determining the dimensions of electrical and magnetic quantities, or determining a coherent and rational system of *units* to which these quantities must be related in order to obtain their *numerical values*, are two quite distinct questions. Unfortunately, it can be said that they have been confused too much, and even that industrial necessities have contributed to this confusion, by making the practical question of *units* predominate over the theoretical question of *dimensions*.[158]

The dimensional indeterminacy of the coefficients k, k', a and λ, leads to several dimensional systems, in each of which the dimensions of the quantities are expressed in terms of L, M, T and the dimensions of one or two coefficients; but unlike the unit systems, these dimensional systems are all correct:

> Each electrical quantity can be expressed in dimensions in several ways as a function of either k, a, λ and k', k and a, λ, k and k', etc., starting from the experimental definition of this quantity, and using the general relationships necessary above. These various expressions will contain nothing arbitrary, and they will be equivalent, which is indispensable, since the same quantity can have only one physical definition, expressible in only one way in terms of the fundamental units.[159]

What's more, none of them correspond to the electrostatic and electromagnetic systems: indeed, the latter "are not independent of the physical nature of the coefficients k, k', a and λ." [160]

Mercadier calls the three relationships previously established between coefficients "transformation formulas for moving from one system to another."[161] He tabulates the

[158] Mercadier 1893a: 875.
[159] Mercadier 1893a: 873.
[160] Mercadier 1893a: 875.
[161] Mercadier 1893a: 874.

dimensions of quantities in five of these systems. He names them after the coefficients used: the one in which quantities are expressed as a function of K - the dimensions of the coefficient in Coulomb's electrostatic formula - is the "Coulomb System," and so on. Three of these systems are "simple", while the other two are "mixed": formulas are obtained by multiplying those of two simple systems. For example, the dimensions of the quantity of electricity in the Coulomb-Ampère system are obtained as follows:[162]

in the Coulomb system:

$$Q_k = n\frac{1}{\sqrt{\mathrm{K}}}\mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-1}\ ,^{163} \qquad \textbf{(162)}$$

in the Ampère system:

$$Q_a = n\frac{1}{\sqrt{\mathrm{A}}}\mathrm{M}^{1/2}\mathrm{L}^{1/2}\ , \qquad \textbf{(163)}$$

and in the Coulomb-Ampère system:

$$Q_{ca} = n\frac{1}{\sqrt{\mathrm{AK}}}\mathrm{ML}^{2}\mathrm{T}^{-1}. \qquad \textbf{(164)}$$

Mercadier thus obtains the following table:

[162] Mercadier 1893a: 873-874.
[163] Where *n* is a dimensionless number; see Mercadier 1893a: 800.

Systèmes rationnels de dimensions.

	Système Coulomb. (expressions en fonction de K)	*Système Ampère.* (expressions en fonction de A)	*Système Laplace.* (expressions en fonction de Λ et K′)
uantité d'électricité......	$Q_k = n\frac{1}{\sqrt{K}}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$	$Q_a = n\sqrt{\frac{1}{A}}M^{\frac{1}{2}}L^{\frac{1}{2}}$	$Q_\lambda = n\frac{\sqrt{K'}}{\Lambda}M^{\frac{1}{2}}L^{\frac{1}{2}}$
otentiel.................	$V_k = n\sqrt{K}M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$	$V_a = n\sqrt{A}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}$	$V_\lambda = n\frac{\Lambda}{\sqrt{K'}}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}$
apacité..................	$C_k = n\frac{1}{K}L$	$C_a = n\frac{1}{A}L^{-1}T^2$	$C_\lambda = n\frac{K'}{\Lambda^2}L^{-1}T^2$

	Système Coulomb. (expressions en fonction de K)	*Système Ampère.* (expressions en fonction de A)	*Système Laplace.* (expressions en fonction de Λ et K′)
Intensité de courant.......	$I_k = n\frac{1}{\sqrt{K}}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}$	$I_a = n\sqrt{\frac{1}{A}}M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$	$I_\lambda = n\frac{\sqrt{K'}}{\Lambda}M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$
Résistance................	$R_k = nKL^{-1}T$	$R_a = nALT^{-1}$	$R_\lambda = n\frac{\Lambda^2}{K'}LT^{-1}$
Auto-induction...........	$L_k = nKL^{-1}T^2$	$L_a = nAL$	$L_\lambda = n\frac{\Lambda^2}{K'}L$
Quantité de magnétisme...	»	»	$\mu_\lambda = n\frac{1}{\sqrt{K'}}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$

Systèmes mixtes.

Système Coulomb-Ampère.	Système Coulomb-Laplace.
$Q^2_{ca} = n\frac{1}{\sqrt{AK}}ML^2T^{-1}$	$Q^2_{c\lambda} = n\frac{\sqrt{K'}}{\Lambda\sqrt{K}}ML^2T^{-1}$
$V^2_{ca} = n\sqrt{AK}\,ML^2T^{-3}$	$V^2_{c\lambda} = n\frac{\Lambda\sqrt{K}}{\sqrt{K'}}ML^2T^{-3}$
$C^2_{ca} = n\frac{1}{AK}T^2$	$C^2_{c\lambda} = n\frac{K'}{\Lambda^2K}T^2$
$I^2_{ca} = n\frac{1}{\sqrt{AK}}ML^2T^{-3}$	$I^2_{c\lambda} = n\frac{\sqrt{K'}}{\Lambda\sqrt{K}}ML^2T^{-3}$
$R^2_{ca} = n\,AK$	$R^2_{c\lambda} = n\frac{\Lambda^2K}{K'}$
$L^2_{c\alpha} = n\,AKT^2$	$L^2_{c\lambda} = n\frac{\Lambda^2K}{K'}T^2$
$\mu^2_{ca} = n\frac{\sqrt{AK}}{\Lambda^2}ML^2T^{-1}$	$\mu^2_{c\lambda} = n\frac{\sqrt{K}}{\Lambda\sqrt{K'}}ML^2T^{-1}$

» Formules de transformation pour passer d'un système aux autres :

$$\frac{K}{A} = L^2T^{-2}; \qquad \frac{KK'}{\Lambda^2} = L^2T^{-2}; \qquad \Lambda^2 = AK'.$$

Table 6.1

These results enabled him to highlight what he called "curious" relationships, such as the fact that the dimensions of the square of a resistor are the product of the dimensions of *a* and *k*. But Mercadier was more interested in relationships that were independent of the system used:

> There are also many other relationships that are independent of those used to form the expressions in the Table, relationships that derive from combinations already known, but which acquire, from the foregoing, a complete generality. Either, in these combinations, the coefficients k, k', a, λ disappear, the mechanical units remaining alone with a numerical factor; or even these units themselves disappear, and the numerical factor remains alone. Thus, the expression RI^2t always represents an energy M L^2T^{-2}(Joule's law): the expressions LI^2 and CV^2 also represent an energy; CR, , $\frac{L}{R}$ $\sqrt{CL}$ represent a time nT; $\frac{CR^2}{L}$ represents a numerical quantity n, and so on. And this, whatever *the nature of k, k', a* and λ.[164]

Given his ideas on the close link between the dimensions of a quantity and its physical nature, Mercadier considers such invariances to be significant: they reveal the nature of things.

This aspect of Mercadier's work recalls some of Maxwell's preoccupations in his Treatise: Maxwell's discussion of systems of units was based on dimensional relations independent of the system under consideration, from which he then derived the electrostatic and electromagnetic systems. Mercadier does the opposite, in a way: he starts from equations specific to the (admittedly different) systems he has just defined, and derives system-independent relations from them. Despite its originality, the definition of dimensional systems is only one stage in Mercadier's reasoning. He then turns to considerations of the units themselves, in a not very original discussion.

[164] Mercadier 1893a: 875.

6.3.3 – Addition of fundamental quantities

In the years following the publication of Vaschy and Mercadier's articles, a number of scientists who shared their opinion that the true dimensions of the coefficients were unknown decided to assume this. They represented the dimensions of these coefficients by symbols, and expressed the formulas for the dimensions of other quantities in terms of these symbols. They therefore treat them in the same way as fundamental quantities - although they don't necessarily accord them the same status.

One of these scientists explicitly refers to Vaschy's 1892 article, discussed above. This was Félix Lucas, who in 1993 suggested using five fundamental quantities: length, time, mass, electric charge and quantity of magnetism (L, T, M,ε and μ).[165] Lucas nevertheless hastens to establish a relationship betweenε and μ, and proposes to take μ as fundamental, attributing toε the dimensions$\mu L^{-1}T$, thus obtaining a set of four fundamental quantities. He explains the reason for this choice as follows:

> If the two theories of magnetism and electricity remained independent of each other, corresponding to two entirely distinct series of phenomena, with no connection between them, the unitsε and μ would have to be considered independent and fundamental to each other. But this is not the case: electromagnetism and electrodynamics show, on the contrary, a similarity of essence between a magnetic mass and an electric mass in motion. We are led to regard magnetic mass μ and electric momentum $LT^{-1}\varepsilon$, the product of electric mass and velocity, as homogeneous.
>
> $$L^{-1}\mu = T^{-1}\varepsilon.$$ [166]

[165] Lucas 1893: 389-392.
[166] Lucas 1893: 390. He gives the dimensions ofε as$\mu L^{-1}T$ p.391.

Lucas seems to have gone astray here, as this is at odds with the $\varepsilon\mu c^2 = 1$ relationship, which implies that ε has $L^{-2}T^2\mu^{-1}$ as its dimensions. Nevertheless, his attempt remains interesting in spirit.

But the first to suggest such an approach was a Briton, W. A. Rücker, professor of mathematics and physics at Leeds, in an 1889 article.[167] Rücker's concerns are reminiscent of those of Mercadier and Vaschy:

> When calculating the dimensions of physical quantities, it's not unusual to obtain indeterminate equations containing two or more unknowns. In such cases, we have to make a conjecture, which usually consists of saying that one of these quantities is an abstract number. In other words, the dimensions of this quantity are eliminated. The dimensions of the units that depend on it, which are subsequently deduced from this conjecture, are clearly artificial, in the sense that they do not necessarily reflect their true relationship to length, mass and time.[168]

Although dimensional analysis can still be used for verification purposes (to ensure that both members of an equation have the same dimensions),[169] this situation is unsatisfactory from a theoretical point of view. Rather than ignoring the dimensions of certain quantities or assuming their exact form, Rücker proposes retaining them in dimensional equations. However, he distinguishes them from the fundamental units L, M and T and calls them "secondary fundamental units". [170]

Although Rücker's reflections are close to those of Mercadier and Vaschy, it cannot be said that he draws his inspiration from them. He doesn't mention them, and refers only

[167] Rücker 1889: 104.
[168] "In the calculation of the dimensions of Physical quantities we not infrequently arrive at indeterminate equations in which two or more unknowns are involved. In such cases an assumption has to be made, and in general that selected is that one of the quantities is an abstract number. In other words the dimensions of that quantity are suppressed. The dimensions of the dependent units which are afterwards deduced from this assumption are evidently artificial, in the sense that they do not necessarily indicate their true relations to length, mass and time." Rücker 1889: 104.
[169] Rücker 1889: 104.
[170] "Secondary fundamental units" Rücker 1889: 106.

to Maxwell. In truth, his approach has an immediate antecedent in the latter's Treatise: the table of dimensions of electromagnetic quantities as a function of M, L, T and e (the electric charge) on the one hand, and M, L, T and m (the quantity of magnetism) on the other.[171] Rücker merely elevates this intermediate calculation to the level of a principle. He believes, however, that this situation is destined to disappear: once the deeper origins of the phenomena in question are known, it will be possible to express the dimensions of any quantity in terms of M, L and T :

> It is likely that, as the mechanical explanation of physical phenomena progresses, the use of units based on simple mechanical conceptions will spread, and that certain quantities now used will either be dispensed with, or considered from another point of view.[172]

Perhaps because it seems simpler to him, the first example Rücker gives concerns the theory of heat, drawing here on a paper by Joseph Bertrand in 1878.[173] Rücker explains:

> A satisfactory expression of the various thermal units remains impossible as long as there is the slightest doubt as to the mechanical definition of temperature.[174]

Rücker's starting point is the equivalence between mechanical work and heat (established by Joule's famous experiments in 1843), which gives :

$$\mathrm{ML^2T^{-2}} = [JMc\theta], \tag{165}$$

where J represents the mechanical equivalent of heat, c the specific heat capacity and θ the temperature. As Rücker assumes that c is numerical,[175] gives

[171] Maxwell 1873: vol.2, chapter 10, 241, see chapter 3, 26.
[172] "It is probable that, as the mechanical explanation of physical phenomena proceeds, the use of units based on simple mechanical conceptions will be extended, that some quantities now employed will be dispensed with or regarded from a different point of view." Rücker 1889: 104.
[173] Bertrand 1878: 916-920. See chapter 7, 249-256.
[174] "The satisfactory expressions of the dimensions of the various thermal units is not possible so long as there is any doubt as to the mechanical definition of temperature." Rücker 1889: 105.
[175] The reasons for this choice are unclear. Rücker states it twice without comment, as if it were self-evident. Rücker 1889: 105-106.

$$[J] = L^2\, T^{-2}\theta^{-1} \qquad \textbf{(166)}$$

Atomistic theory would allow us to go further if it proved correct:

> If we were sufficiently certain that the analogy with gas theory could be extended to all cases, it would follow that temperature depends only on the average energy of a particular type (for gases, the average translational energy) that molecules have. Since the average value of a quantity has the same dimensions as that quantity itself, temperature could therefore be considered not only as measurable by the average translational energy of a molecule, or of a standard number of molecules, but even as being that average energy. The dimensions of temperature would therefore be . ML^2T^{-2} [176]

Deeming the question still uncertain, Rücker maintains an independent unit Θ for temperature. And he gives a table of dimensions for thermal quantities, taking Θ into account.[177] Vaschy adopted this table in his 1890 treatise, as he endorsed the need for a fundamental unit of temperature:

> As for the Θ unit of temperature, it could not be attached to the L, M, T system, and constitutes, in our present state of knowledge, a fourth fundamental unit.[178]

However, Vaschy reduces heat to entirely mechanical dimensions, again on the basis of its equivalence with work:

$$[ch] = [W] = [F]L = ML^2T^{-2} \qquad \textbf{(167)}$$

Bertrand did the same in 1878.[179]

Rücker then applies the same principle to the field of electromagnetism, and proposes to retain in the dimension formulas symbols representing the dimensions of

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[176] "If it were sufficiently certain that the analogy offered by the theory of gases might be extended to all cases, it would follow that temperature depended only on mean energy of a particular kind (in gases the mean energy of translation) possessed by the molecules. The mean value of a quantity is of the same dimensions as the quantity itself, and thus temperature might be regarded not only as being measured by, but as being the mean energy of translation of a molecule, or of a standard number of molecules. The dimensions of temperature would thus be [M $L^{(2)}$ $T^{(-)(2)}$]." Rücker 1889: 105.
[177] Rücker 1889: 106.
[178] Vaschy 1890: 7.
[179] Bertrand 1878: 919.

inductive power (K) and permeability (μ).[180] This would give us five fundamental quantities.

A final interesting example comes from E. Brylinski, a telegraph engineer. This was in response to Malagoli's suggestion that the fundamental quantities should be the quantity of electricity *Q*, the quantity of *magnetism m* and the work (or energy) *W*.[181] Malagoli justified this choice by the simplicity of the resulting dimensional formulas and by his energetic conceptions. Brylinski, while approving of the latter, objected to Malagoli's suggestion on the grounds that his formulas were wrong. He shares Mercadier and Vaschy's view that a physical quantity has only one valid dimensional formula - and that this does not correspond to that of the electrostatic system. Malagoli's reasoning implies that his new fundamental units Q and m have the dimensions :

$$Q = L^{1/2}M^{1/2} \text{ and . } \quad m = L^{3/2}M^{1/2}T^{-1} \tag{168}$$

Brylinski points out that these formulas are true only in the electrostatic system.[182] However, "nothing in the present state of science allows us to express *K* [the coefficient in Coulomb's law for electrostatics] as a function of L, M, T."[183] Brylinski goes further than Mercadier and Vaschy in that he questions the mechanistic credo, stating that nothing allows "even the assertion that K can be expressed in terms of these units [L, M, T]. The same uncertainty reigns on this point as on the dimensions of temperature in heat."[184] Brylinski defends the idea that, in any case, the real dimensions of *K* are unknown, and it

[180] Rücker 1889: 108-109.
[181] See chapter 4. Malagoli actually uses the symbols Q_e, $Q_m$and W, respectively. Malagoli 1897: 535-539.
[182] Brylinski 1897: 61.
[183] Brylinski 1897: 61.
[184] Brylinski 1897: 61.

must therefore be retained as an independent fundamental quantity, including in the system proposed by Malagoli.[185]

6.3.4 - Attempts to determine the dimensions of coefficients

Finally, some scientists set out to determine the dimensions of the coefficients, not through experiments as Vaschy and Mercadier had suggested, but through theoretical reasoning. In Ireland, in 1889 FitzGerald proposed an intermediate system of units between the electrostatic and electromagnetic systems.[186] Having noted that the former system assumes that electrical capacitance has no dimensions, and the latter that magnetic capacitance has none, FitzGerald suggested giving these two quantities the same dimensions: $[k] = [k']$

> If we take a system for which the dimensions of these quantities are the same, and are those of a slowness, i.e. the inverse of a speed, $\left[\frac{\mathrm{T}}{\mathrm{L}}\right]$, the two systems become identical as regards dimensions, and differ only by a numerical coefficient, exactly as centimetres differ from kilometers. [187]

FitzGerald's main motivation for suggesting such a system seems to be its symmetry. He adds:

> There's something natural about this result that justifies the assumption that these inductive abilities are truly of the sluggish nature.[188]

[185] Brylinski 1897: 61.
[186] Fitzgerald 1889: 323.
[187] "If we take a system in which the dimensions of both these quantities are the same, and of the dimensions of a slowness, i.e. the inverse of a velocity, $\left[\frac{\mathrm{T}}{\mathrm{L}}\right]$, the two systems become identical as regards dimensions, and differ only by a numerical coefficient just as centimetres and kilometres do." Fitzgerald 1889: 323.
[188] "There seems a naturalness in this result that justifies the assumption that these inductive capacities are really of the nature of a slowness." Fitzgerald 1889: 323.

There is no evidence to suggest that FitzGerald was directly aware of Mercadier and Vaschy's ideas. He may well have been inspired by the work of Heinrich Hertz (1857-1894), who had just produced electromagnetic waves through rapid electrical oscillations. A few years earlier, in 1884, Hertz had set out to remove an asymmetry in electromagnetic theory.[189] He wrote Maxwell's equations in a symmetrical form, permuting the electric and magnetic fields. This type of writing made it possible to eliminate potentials, deemed less physical than fields. FitzGerald also used it.

Another work of the same type is that of a certain P. Joubin, published in 1896 in the *Journal de physique*.[190] Joubin attempts to determine the dimensions of the inverses of the coefficients *k* and *k'*, the electric and magnetic inductive powers, which he represents by *K* and *K'*. Although he does not take it for granted that *K* and *K'* "can be expressed exclusively by means of mechanical units,"[191] he accepts this property as a postulate. He continues:

> If, according to our postulate, these quantities [*K* and *K'*], are of no other nature than those ordinarily encountered in rational mechanics, their dimensions are necessarily whole in L, M, T."[192]

Although nothing in his discussion indicates where he got this opinion, we can guess that he found that all usual mechanical quantities have integer dimensions. Let's not forget that in 1879, Everett in Great Britain judged the dimensions of electrical units to be simpler if

[189] Buchwald 1994: 200-202.
[190] Joubin 1896a: 188-191; also: Joubin 1896b: 398-401.
[191] Joubin 1896a: 188.
[192] Joubin 1896a: 189.

they were integer - a criterion he satisfied by taking length, time and density as fundamental quantities.[193] Malagoli also used this same criterion in 1897.[194]

Joubin considers the dimensions of electrical and magnetic quantities in terms of L, M, T and K. Clearly, they correspond to Mercadier's "Coulomb system":

Quantity of electricity:	$Q_k = n\,\mathrm{K}^{-1/2}\mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-1}$	**(169 a)**
Potential :	$V_k = n\,\mathrm{K}^{-1}\quad \mathrm{L}$	**(170 b)**
Capacity :	$C_k = n\,\mathrm{K}^{-1}\quad \mathrm{L}$	**(171 c)**
Current :	$I_k = n\,\mathrm{K}^{-1/2}\mathrm{M}^{1/2}\mathrm{L}^{3/2}\mathrm{T}^{-2}$	**(172 d)**
Resistance:	$R_k = n\,\mathrm{K}\quad \mathrm{L}^{-1}\mathrm{T}$	**(173 e)**
Self-induction:	$L_k = n\,\mathrm{K}\quad \mathrm{L}^{-1}\mathrm{T}^{2}$	**(174 f)**
Quantity of magnetism :	$\mathcal{M}_k = \mathrm{K}\quad \mathrm{L}^{-1}\mathrm{T}^{2}$	**(175 g)**

Table 6.2

Joubin notes the following regularities: mass always appears with exponent ½, and time with an integer exponent. If mass does not appear, length and K appear with an integer exponent, otherwise they appear with a fractional exponent. Since K must have integer exponents for L, M and T, Joubin concludes that its dimensions are odd in L and M, and even in T. [195]

[193]Everett 1879 : 134.

[194] See chapter 4, 147.

[195] Joubin doesn't spell out his reasoning, but it's easy to reconstruct. For dimensions only: $K = M^{(a)} L^{(b)} T^{(c)}$where a, b and c must be integers. For Z any electrical or magnetic quantity: $Z = M^{(d)} L^{(e)} T^{(f)}$where d, e and f must also be integers, since Joubin believes that any quantity expressible in terms of the fundamental mechanical quantities has integer exponents of dimensions in the latter. None of these six exponents is known, but the following four are known: $Z = K^{(x)} M^{(g)} L^{(h)} T^{(i)}$. Clearly: $K^x = (M^{(a)} L^{(b)} T^{(c)})^x = M^{(ax)} L^{(bx)} T^{(cx)}$. Substituting this expression into the second formula for Z: $Z = M^{(ax+g)} L^{(bx+h)} T^{(cx+i)}$. Comparing this result with our first expression for Z, $Z = M^{(d)} L^{(e)} T^{(f)}$, we obtain :

For M: $ax + g = d$ For L: $bx + h = e$ For T: $cx + i = f$

In the case of M, we know that x and g are always "fractional" at the same time (more precisely, are multiples of ½). For d to be an integer, a must therefore be odd. Similar reasoning implies that b must also be odd. On the other hand, for T, we know that i is always an integer. For f to be an integer, then, c must be even.

Since the dimensions of the product KK' are the square of the inverse of a velocity, the dimensions of K' are also odd in L and M and even in T.[196] And since the product $K\,K'$ does not contain mass, the exponents for mass must be of the same absolute value for K and K', and of opposite sign. Joubin also thinks that their absolute value must be 1, because "all the quantities of mechanics [contain] mass only to the first power."[197] For the purposes of his demonstration, he takes 1 as the exponent of mass for K', and summarizes his conclusions as follows:[198]

$$[K] = \mathrm{M}^{-1}\mathrm{L}^{2p-1}\mathrm{T}^{2q} \qquad \textbf{(176)}$$
$$[K'] = \mathrm{M}\,\mathrm{L}^{-(2p+1)}\,\mathrm{T}^{-(2q-2)} \qquad \textbf{(177)}$$

He then proposes reasonable values for the dimension exponents: ± 1 or ± 3 for L, and 0 or ± 2 for T. This assumption implies twelve possible dimension formulas for K, and as many for K',[199] but Joubin notes that only half are truly distinct, the other half being obtained by "exchanging K and K'." [200]

K K' [201]

[196] If we put $[K'] = M^{j}L^{k}T^{l}$, we get $K\,K' = \mathrm{M}^{a+j}\ \mathrm{L}^{b+k}\mathrm{T}^{c+l} = \mathrm{M}^0\mathrm{L}^{-2}\mathrm{T}^2$, and so the sums a + j, b + k and c + l are all even. Since a and b are odd, j and k must also be odd, while since c is even, l must be even. So we have: $[K'] = M^{j}L^{k}T^{l}$ with j and k odd and l even.

[197] Joubin 1896a: 189.

[198] If we express the odd character of b by 2 p-1 , given that $K\,K' = \mathrm{L}^{(-2)}\,\mathrm{T}^{(2)}$, so that b + k = -2, we have for k:

b + k = - 2 hence b = - 2 -k

and b = 2 p - 1 so : - 2 - k = 2 p -1 hence k = - (2 p + 1)

Similarly, if we express the evenness of c by c = 2 q, and given that K K' = $\mathrm{L}^{(-2)}\,\mathrm{T}^{(2)}$then c + l = 2 :

c + l = 2 hence c = 2 - l

and c = 2 q therefore : 2 - l = 2 q hence l = - (2 q - 2)

[199] We have 4 distinct possibilities for L and 3 for T. Consequently, there are, on the one hand, four possibilities for 2 p -1 and three for 2q, so twelve in all for K, and, on the other hand, four for - (2p+1) and three for - (2q - 2) so twelve in all for K'.

[200] In fact, by taking a = 1 and j = -1 instead of the opposite.

[201] The exponents of L and T for K' are obtained as follows:

$[K] = M^{a}L^{b}T^{c}$ $[K'] = M^{j}L^{k}T^{l}$

For L : $b + k = -2$, so when :

$b = -1$, ; $k = -1$ $b = 1$, $k = -3$; $b = -3$, ; $k = 1$ $b = 3$, $k = -5$

For T : $c + l = 2$, so when :

(1)	$\mathbf{M^{-1}L^{-1}T^{2}}$	$\mathbf{ML^{-1}}$
(2)	$\mathbf{M^{-1}L^{1}T^{2}}$	$\mathbf{ML^{-3}}$
(3)	$\mathbf{M^{-1}L^{-3}T^{2}}$	$\mathbf{ML^{1}}$
(4)	$M^{-1}L^{3}T^{2}$	ML^{-5}
(5)	$M^{-1}L^{-1}T^{2}$	$ML^{-1}T^{4}$
(6)	$M^{-1}L^{1}T^{2}$	$ML^{-3}T^{4}$
(7)	$M^{-1}L^{-3}T^{2}$	$ML^{1}T^{4}$
(8)	$M^{-1}L^{3}T^{2}$	$ML^{-5}T^{4}$
(9)	$\mathbf{M^{-1}L^{-1}}$	$\mathbf{ML^{-1}T^{2}}$
(10)	$\mathbf{M^{-1}L^{1}}$	$\mathbf{ML^{-3}T^{2}}$
(11)	$\mathbf{M^{-1}L^{-3}}$	$\mathbf{ML^{1}T^{2}}$
(12)	$M^{-1}L^{3}$	$ML^{-5}T^{2}$

Table 6.3

At this point, Joubin's reasoning becomes intuitive: it's by inspecting these dimension formulas that he decides which ones make physical sense.[202] He states without explanation that of these twelve combinations "*only one gives both for K and K' dimensions having a mechanical meaning*" :[203]

$$[K] = \mathrm{M^{-1}LT^{2}}\,, \qquad [K'] = \mathrm{ML^{-3}}\,. \tag{178}$$

Joubin deduced that *K*' is a density and *K* a "compressibility coefficient," since its dimensions are the inverse of those of an elasticity coefficient.[204] He notes that these are the quantities involved in the equation for the propagation velocity of a wave in an elastic medium $v = \sqrt{\dfrac{e}{d}}$ where *e* is the compressibility coefficient and *d* the density of the medium. It's not clear, however, whether this is an *a posteriori* interpretation on his part, or whether

$c = 0,\quad l = 2;\quad c = 2\,,;l = 0 \qquad c = -2,\ l = 4$

[202] Given his previous remarks, you'd expect him to have eliminated from the outset formulas with exponents greater than 3, i.e. those containing either T^{4}or L^{-5}, which we have therefore italicized in the table. This reduces the number of possibilities by half. It's not clear what criteria Joubin uses to make his selection.

[203] Joubin 1896a: 189.

[204] From $\dfrac{F}{A} = E\dfrac{\Delta L}{L}$ you can find $\dfrac{\mathrm{MLT^{-2}}}{\mathrm{L^{2}}} = [E]\dfrac{\mathrm{L}}{\mathrm{L}}$ or .$[E] = \mathrm{ML^{-1}T^{-2}}$

this fact played an essential role in his choice of combination (28), especially as he admits to having expected to find this result:

> These are the values that we would have been led to give to *K* and *K*' to represent the speed of propagation of an electromagnetic wave.[205]

Once the formulas for the dimensions of *K* and *K*' had been obtained, Joubin derived the dimensions of the other electrical and magnetic quantities.

The impact of Mercadier and Vaschy's work is therefore complex. In any case, the system they prefer - the electromagnetic system - is the one that has long since prevailed in Great Britain and among French telegraphists, and which had also just been adopted two years earlier at the 1881 International Electricity Congress. In this sense, their work is an afterthought, and its impact cannot be judged by the use of one system over another in practice. We must therefore confine ourselves, as we have done, to the ideas expressed in publications subsequent to theirs, and it is not always clear to what extent the latter are directly inspired or not by the two engineers' ideas. No one presents arguments - at least not directly in response to Mercadier's articles - against the notion, unusual at the time and even provocative, that they set up as a fundamental principle: that a physical quantity can only have a single dimensional formula. This notion explicitly stems from Mercadier and Vaschy's notion that the dimensions of a physical quantity reveal its very nature. In this respect, their work constitutes a return to the roots of the concept of dimension: it is linked

[205] Joubin 1896a: 189. The question arises as to what we get if we use a dimensional system other than the "Coulomb system", which involves only K. Unfortunately, none of the other systems presented by Mercadier involves only K'. Unfortunately, none of the other systems presented by Mercadier involve only K'. Nevertheless, we can see that in all cases, when the dimension exponent of L is fractional, so is that of M, and vice versa. Moreover, T always appears with an integer exponent. Finally, the only other system presented by Mercadier that contains only one coefficient, i.e. the Ampère system with A, seems to lend itself to a similar analysis, since A happens to have a fractional exponent when L and M do, and vice versa.

to the notion of homogeneity in its etymological sense, which Fourier's work, by redefining this notion in the context of conversion between systems of units, had eclipsed. From an epistemological point of view, this is a realistic attitude to dimension formulas.

Chapter 7: Rayleigh's "method of dimensions"

7. 1 - Development of the "method of dimensions"

In the years1870, dimensional analysis began to be used in contexts other than that of changes of units. In particular, John William Strutt (1842-1919), later Lord Rayleigh, showed how it could be used to determine the form of the relationship between a physical quantity and the variables on which it depends.[1] After Fourier's redefinition of dimensions, this was undoubtedly the first case where dimensional considerations were used to obtain original physical equations, and not just for verification or conversion purposes. In 1816, Poisson had used the principle of homogeneity to find a relationship between the speed of waves at the surface of a fluid and other quantities, but his concept of homogeneity was limited to geometric dimensions.[2]

Rayleigh was one of the great masters of English physics in the second half of th e 19th century, both in theory and experiment. At the start of his career, in the 1870s, he concentrated on acoustics and optics, before moving on to electromagnetism and the study of gases. Among his most famous contributions are his explanation of the blue color of the sky and, above all, his acoustical treatise *Theory of sound* (1877-1878) - whose influence was so considerable that even today it remains a reference work on the subject - and his discovery of argon in 1895. His experimental research was characterized by a

[1] Rayleigh 1871: 107-120 & 274 -279.
[2] See chapter 1, 30-33.

preoccupation with precise measurements. When he succeeded Maxwell as *Cavendish Professor* at Cambridge from 1879 to 1884, he directed a program of this nature concerning electrical standards. Similarly, it was a program of precision measurements of gas density, conducted in the 1880s, that led him to postulate the existence of an inert gas in the atmosphere and to isolate argon.[3]

Rayleigh's first dimensional reasoning appears in his famous 1871 memoir, in which he explains the blue color of the sky by the diffusion phenomenon that now bears his name. Here, he applies dimensional analysis to two specific problems.

The first is to determine an equation expressing the relationship between the amplitude of an incident light ray and that of a ray after interaction with material particles in the atmosphere. Rayleigh took it for granted that light in the daytime sky reaches us after its trajectory has been altered by material particles suspended in the atmosphere. He sees light as a transverse wave propagating through the ether, to which we can apply the model of an elastic solid. The presence of material particles exerts forces on the ether that affect the propagation of this wave. He points out that, to explain the color of the sky, the size of material particles must be smaller than the wavelength of light. He generally uses the term "scattered" in reference to this interaction. [4]

Rayleigh denotes by i the ratio he sets out to determine (between the amplitude of an incident light ray and that of a refracted ray) and assumes that it depends at most on the following quantities:

> T: the volume of the particle obstructing the beam.
> r: the distance between the point in space of interest (i.e. where the intensity of the refracted ray is measured), and the particle in question.

[3] Darrigol 2005: 208; Porter 1994: 576-577.
[4] Rayleigh 1871: 107-108.

λ wavelength of the light in question.
b: the propagation speed of this light (in this case in air)
D: initial density
D': density after interaction.

Rayleigh takes the usual fundamental quantities, i.e. length, time and mass, and notes that T, r and λ depend on length alone, b on length and time, and D and D' on mass (and length). Since the quantity whose relationship he is looking for is a ratio of homogeneous quantities, it is of course dimensionless.

Rayleigh then considers each fundamental quantity one after the other, starting with the one that intervenes least, mass. D and D' must appear in his formula as a quotient, since they are the only quantities that depend on mass. Rayleigh then considers time. Since only b depends on it, the ratio i cannot contain it.

It remains to determine the relationship between i on the one hand, and T, r and λ on the other, considering homogeneity with respect to length. Rayleigh remarks at this point that "from what we know of the dynamics of the subject, we may be sure that i is proportional to T and inversely proportional to r." [5] All that remains is to determine the dependence of i on λ. Clearly, homogeneity with respect to length implies

$$i \propto \frac{T}{\lambda^2 r}. \qquad \textbf{(179)}$$

Rayleigh thus obtains the following law:

> When light is scattered by particles that are very small in comparison to all wavelengths, the ratio of the amplitudes of the vibrations of the scattered light to those of the incident light varies as the inverse of the square of the wavelength, and the intensities of the lights themselves vary as the inverse of the fourth power. [6]

[5] "From what we know of the dynamics of the question, we may be sure that i varies directly as T and inversely as r." Rayleigh 1871: 110-111.
[6] "When light is scattered by particles which are very small compared with any of the wave-lengths, the ratio of the amplitudes of the vibrations of the scattered and incident light varies inversely as the square of the wave-length, and the intensity of the lights themselves as the inverse fourth power." Rayleigh 1871: 111.

Rayleigh then set out to determine "the mathematical expression of the perturbation which propagates in a given direction from a small particle struck by a ray of light ."[7] Dimensional analysis is insufficient here, as direction involves dimensionless quantities such as angle cosines.

In the same work, Rayleigh uses dimensional reasoning for a second case.[8] The aim is to determine how the reflection coefficient of light on a thin plate varies as a function of the wavelength of the light. Rayleigh takes as his only variables this wavelength λ and the thickness δ of the blade. The only fundamental quantity he considers is length, and, as in the previous demonstration, he gives himself an additional relationship: "The reflected vibration is necessarily proportional to δ."[9] Dimensional homogeneity then implies that the reflection coefficient is inversely proportional to λ. Rayleigh concludes that the model hitherto used for the refraction of light by the earth's atmosphere, which was based on the analogy with reflection by thin plates, is at odds with the one he himself proposes:

> Since Newton, many physicists have assumed that light from the sky is reflected by thin blades, and that its color is first-order blue in Newton's scale. Such a conception is fundamentally different from the one we adopt in this article.[10]

As early as 1871, Rayleigh was able to obtain original results from dimensional reasoning. But there is no real method here, nor any reflection on the possible limits of such demonstrations. It wasn't until 1877, in his *Theory of sound*, that Rayleigh turned the

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[7] "The mathematical expression for the disturbance propagated in any direction from a small particle which a beam of light strikes," Rayleigh 1871: 111.
[8] Rayleigh 1871: 115.
[9] "The reflected vibration necessarily varies asδ ." Rayleigh 1871: 115.
[10] "By many physicists, fromNewton downwards, the light of the sky has been supposed to be reflected from thin plates, and the colour to be the blue of the first order inNewton 's scale. Such a view is fundamentally different from the one adopted in this paper." Rayleigh 1871: 115.

type of reasoning he had developed in 1871 into a real method.[11] The first case treated concerns the vibrations of a metal wire (such as an instrument string). The situation is as follows: the wire, under tension, is fixed at both ends, and a mass is attached to its middle.[12] The mass is then moved from its equilibrium position. Rayleigh illustrates the situation with the following figure:[13]

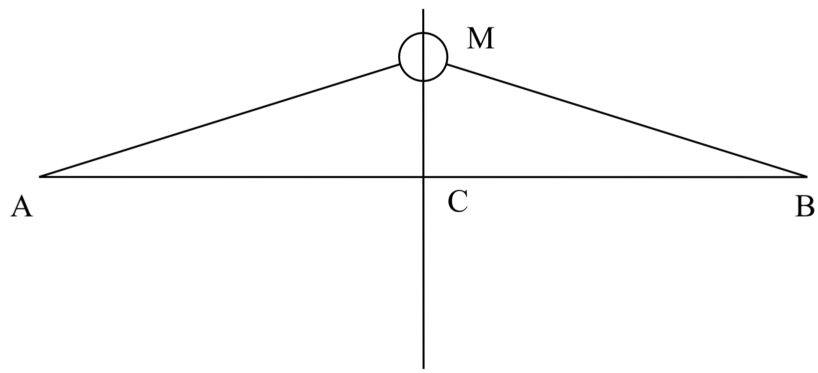

Fig.1

The wire in its initial state is represented by AB, and once the mass has been moved by AMB. C represents the equilibrium position of the mass, and M the mass itself once moved. The variables of interest here are :

M: mass
T: wire tension
a: length AC or BC (half the length of the wire in its initial state)
τ : the period at which the system oscillates when the mass is released.

[11] Cf. Macagno 1971: 391-402, Roche 1998: 208-211.
[12] Rayleigh 1877: vol.1, 46-47.
[13] Rayleigh 1877: vol.1, 46.

Rayleigh considers the case of small oscillations; he assumes that the extension of the wire in the AMB position is negligible, so that the tension can be taken as constant during the oscillations.

Rayleigh begins by deriving

$$\tau = 2\pi \div \sqrt{\frac{2T}{aM}} \qquad \textbf{(180)}$$

from the equation of motion. He then sets out to justify this relationship on the basis of dimensional considerations. His only assumption is that τ must depend solely on the variables a, T, M: $\tau = f\ (a, M, T)$. At this point, the theoretical argument he offers to justify his use of dimensions is in line with Fourier's concerns:

> This equation [$\tau =$ f (a, M, T)] must retain its unchanged form whatever the fundamental units by means of which the four quantities are expressed numerically, as is evident when we consider that in deriving it we have made no assumptions as to the magnitude [*magnitude*] of these units. [14]

Rayleigh insists on this aspect in his demonstration. We find the following expressions: "so that [the equation in question] is independent of the unit of length" or "so as to ensure the independence of the unit of mass."[15] As we have seen, in his 1871 paper Rayleigh referred more directly to the dimensions of quantities. We now find him much closer to Fourier in his approach. It's likely that this change is in response to a desire on Rayleigh's part to be more explicit and clear.

Rayleigh first shows how to obtain the desired relationship between τ and the other variables by appealing to his readers' common sense. He explains that since voltage is the

[14] "This equation [$\tau = f\,(a, M, T)$] must retain its form unchanged, whatever may be the fundamental units by means of which the four quantities are numerically expressed, as is evident, when it is considered that in deriving it no assumptions would be made as to the magnitudes of those units." Rayleigh 1877: vol.1, 47.
[15] "In order that (3) [the relation under discussion] may be independent of the unit of length" or "in order to secure independence of the unit of mass." Rayleigh 1877: vol.1, 47.

only one of the three variables that depends on the unit of time, and since voltage T has dimensions , MLT^{-2} [16] we must have :

$$\tau \propto T^{-1/2}. \tag{181}$$

Since the dependence of τ on T is known, Rayleigh focuses on its dependence on a, the only variable other than voltage T that depends on the unit of length. Since τ cannot depend on the unit of length, and since voltage T has dimensions MLT^{-2} , we must have :

$$\tau \propto T^{-1/2} a^{1/2}. \tag{182}$$

Continuing the same type of reasoning for the mass, he finally obtains :

$$\tau \propto T^{-1/2} M^{1/2} a^{1/2}. \tag{183}$$

Having illustrated his reasoning with this concrete example, Rayleigh then presents it more formally, developing a method. His starting point is the monomial expression

$$\tau \propto T^{x} M^{y} a^{r} \tag{184}$$

and determines the exponents of this formula. He then obtains three linear equations for these exponents, in accordance with the three steps described above. Thus, for time we have :

$$1 = -2x \tag{185}$$

because the dimension exponents of τ, T, M, and a are $1, -2$, 0, 0, respectively. Similarly, we obtain for length :

$$0 = x + z, \tag{186}$$

and for the mass :

$$0 = x + y. \tag{187}$$

[16] Rayleigh doesn't yet use this notation in this part of his work, but he then makes constant use of it in the other cases he discusses.

This gives us three equations with three unknowns, which we simply need to solve to obtain the powers x, y and z at which the variables appear in the equation we're interested in.

Rayleigh's approach is reminiscent of that used by Fourier to *check* the homogeneity of an equation.[17] But instead of giving ourselves x, y and z and checking that the exponents of the two members of an equation match, we treat these x, y and z as unknowns.

In a later passage of his *Theory of sound*, Rayleigh again uses the dimensional method, in the case of vibrations of a string of given linear mass and tension - a case of obvious importance in music.[18] He again illustrates the power of the dimensional method by re-deriving three fundamental relations he has just outlined:

> - For a given tension T_1, the vibration period of a string varies proportionally to its length.
>
> - For a string of given length l, the vibration period is inversely proportional to the square root of the tension.
>
> - For a given voltage and length, the vibration period is proportional to the square root of the linear specific mass ρ.

The dimensions of the three variables are as follows:

$$[T_1] = \mathrm{MLT}^{-2},\ [\rho] = \mathrm{ML}^{-1},\ [l] = \mathrm{L}\ . \qquad \textbf{(188)}$$

If we pose

$$\tau \propto T_1^x\, \rho^y\, l^z \qquad \textbf{(189)}$$

we obtain the three equations :

time :	$1 = -2x + 0y + 0z$
length :	$0 = x - y + z$
mass :	$0 = x + y + 0z$

[17] See chapter 1.
[18] Rayleigh 1877: vol.1, 139.

Solving this system, we get :

$$x = -1/2, \qquad y = 1/2, \qquad z = 1. \tag{190}$$

The relationship between the period and the three variables is therefore as follows:

$$\tau \propto T_1^{-1/2} \rho^{1/2} l^{1/2} \tag{191}$$

which sums up the three principles outlined above in a single equation.

In the second volume of his work, Rayleigh uses dimensional analysis to study the propagation of sound. He thus recovers a result previously obtained by Newton. He assumes that the speed of sound depends only on pressure p and density ρ. Knowing that these quantities have dimensions :

$$[p] = \mathrm{ML^{-1}T^{-2}} \qquad [\rho] = \mathrm{ML^{-3}}. \tag{192}$$

Rayleigh deduced that the speed of sound is proportional to . $\sqrt{\frac{p}{\rho}}$ [19]

Rayleigh's final case is that of the reflection of sound by an obstacle. More precisely, he considers plane waves travelling in a fluid, and encountering a region (the obstacle), of a density and compressibility different from that of the medium, and of arbitrary shape.[20] He is interested in the quotient of the amplitude of the waves reflected by this obstacle to the amplitude of the incident waves, and demonstrates that this quotient is inversely proportional to the square of the wavelength. This situation is therefore analogous to that of scattering discussed above, but this time for sound rather than light

[19] Rayleigh 1877: vol.2, 18. Here we follow the notation used by Rayleigh to express dimension formulae.
[20] Rayleigh 1877: vol.2, 135.

waves. The demonstration differs in detail, however. There are five quantities on which this quotient may depend:[21]

T: the volume of the obstacle, such that its density and compressibility constant vary significantly compared to the rest of the medium.
r: distance between the point in question and the obstacle
λ : wavelength
a: the speed of sound
σ: normal density of the medium

Rayleigh does not bother to specify their dimensions, but they are of course as follows:

$\sigma : \mathrm{ML}^{-3}$ $\qquad$ $\mathrm{r : L}$ $\qquad$ $\lambda : \mathrm{L}$ $\qquad$ $a : \mathrm{LT}^{-1}$ $\qquad$ $\mathrm{T : L}^3$

Unlike the previous cases, this time we have five variables, and therefore five unknowns, whereas dimensional considerations provide us with only three equations. Rayleigh does not comment on this difficulty. He simply points out that, since the quantity he is interested in is the quotient of two amplitudes, which must therefore be dimensionless, neither σ nor a can intervene in the formula expressing this quotient. It will be remembered that in his 1871 paper on the scattering of light, he considered two densities of medium, and assumed that their quotient had to intervene, rather than not taking density into account at all (as he must do here, since he has only one density). If we pose (designating "q" as the quotient of the amplitudes):

$$q \propto \sigma^v r^w \lambda^x a^y T^z \tag{193}$$

we obtain the following three equations:

$$\text{Mass :} \quad 0 = v \tag{194 a}$$
$$\text{Time :} \quad 0 = y \tag{194 b}$$
$$\text{Distance :} \quad 0 = w + x + 3z \tag{194 c}$$

[21] Rayleigh 1877: vol.2, 138.

Solving this problem from dimensional considerations alone is impossible. Rayleigh therefore appealed to relations he had previously obtained by other methods.[22] He knows that q must vary proportionally to T, and inversely proportionally to r. So he already knows that $w = -1$ and $z = 1$. He can then obtain $x = -2$ from the dimensional equation that corresponds to the distance. The final result is :

$$q \propto r^{-1} \lambda^{-2} T . \qquad \textbf{(195)}$$

In these latter examples, the dimensional method makes it possible to determine or specify the relationship between a given physical quantity and the variables on which it depends. At first glance, this seems extraordinary and promising, even offering the possibility of discovering the relationship between all physical quantities without having to resort to experiment. But we must be wary of such an interpretation. While the method does offer the possibility of discovering novel relationships, it cannot be used in all cases. Without additional information, it can only be applied to cases where there are at most three variables, since only three equations can be obtained from considerations of homogeneity with respect to length, time and mass. What's more, it doesn't take into account non-dimensional variables. Above all, it can in no way replace experiments, for a reason that Rayleigh strongly emphasizes: it requires us to know from the outset all the variables on which the quantity of interest is likely to depend.[23] It also requires that we know the dimensions of all these variables. While this last condition does not seem to be a problem in the examples we have dealt with so far, all of which come from mechanics, the situation is quite different in other branches of physics, notably electromagnetism, where,

[22] Rayleigh 1877: vol.2, 138.
[23] Rayleigh 1877: vol.1, 47.

as we have seen, the dimensions of the same quantity vary from one system to another - and are sometimes not unanimously accepted in a given system.

The need to know from the outset all the variables on which the quantity of interest depends is particularly problematic, because if one or more of these variables are neglected, the method described by Rayleigh generally leads to erroneous results. Thus, he does not hesitate to describe it as "dangerous", despite its power and fruitfulness:

> Because of the need for a complete enumeration of all the quantities on which the result may depend, the “method of dimensions” is somewhat dangerous; but when used with sufficient care it is of undeniable power and value. [24]

Rayleigh devotes the final pages of his book to dimensional analysis. Although his stated aim is to summarize the issue and develop the principles on which the method is based, he confines himself to giving a brief example of a general nature, and quickly moves on to the link between dimensions and dynamic similarity, which concerns the comparison of the behavior of similar systems within a change of scale. We describe his discussion in detail in Chapter 8, devoted to similarity.[25]

Rayleigh's work occupies an important place in the literature on dimensional analysis in the 19th century. Roche sees this contribution as the founding act of what he calls "*dimensional exploration*", which is the subject of one of the two chapters he devotes to dimensional analysis:

> In the *Theory of Sound* and in several subsequent publications, [Rayleigh] showed how dimensional analysis can become a strategy for discovery. In doing so, he

[24] "From a necessity of a complete enumeration of all the quantities on which the required result may depend, the method of dimensions is somewhat dangerous; but when used with proper care it is unquestionably of great power and value." Rayleigh 1877: vol.1, 47.
[25] Rayleigh 1877: vol.2, 287-289.

> seems to have introduced a largely innovative method of investigation in mathematical physics, which I shall call 'dimensional exploration.'[26]

Enzo Macagno goes even further, and sees in *Theory of sound* the first significant use of Fourier's contributions: "After Fourier, there were no important developments in dimensional analysis for half a century"; Rayleigh's treatise "marks the beginnings of the application of the principles enunciated by Fourier more than fifty years earlier."[27] Note that for Macagno, Rayleigh's considerations are merely an "application" of principles already given by Fourier:

> The fact that Rayleigh developed a method, while Fourier established a principle, reflects the two scientists' different attitudes to physical dimensions. Rayleigh never gave his method a logical-deductive structure, but made good use of it as a research tool.[28]

While it is true that Rayleigh proposed a method rather than a principle, limiting his contributions to a practical application of what Fourier had presented seems to do his contributions no justice. Fourier had not suggested that dimensions could be used to discover relationships between quantities. As Roche notes, Rayleigh's method is a discovery strategy. What's more, the way Rayleigh presents his method is no less structured than the way Fourier described the use of his principle for verification purposes, or to determine conversion factors.

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[26] "In his *Theory of Sound* and in many subsequent publications [Rayleigh] showed how dimensional analysis could become a strategy of discovery. In so doing he seems to have introduced a substantially new method of enquiry to mathematical physics which I shall call 'dimensional exploration'," Roche 1998: 208.

[27] "After Fourier, no important development in dimensional analysis took place for half a century," "marks the beginning of the application of principles enunciated by Fourier more than fifty years before," Macagno 1971: 395.

[28] "The fact that Rayleigh developed a method, while Fourier established a principle, reflects the different attitudes of the two men of science with respect to physical dimensions. Rayleigh never gave a logico-deductive structure to his method, but made extensive and good use of it as a research tool," Macagno 1971: 395.

More than a decade later, in 1892, Rayleigh used the dimensional method in a paper devoted to fluid flow in tubes.[29] This time, he set out to understand the relationships established in 1883 by Osborne Reynolds (1842-1912) for this type of flow.[30] Rayleigh attributed his results to Reynolds' reasoning by similitude, but there is no evidence of this in Reynolds' work, which was more empirical in approach.[31]

The variables to be taken into account are as follows: [32]

c: tube diameter
ρ: fluid density
w: fluid velocity
μ : coefficient of viscosity (or) $v = \dfrac{\mu}{\rho}$

The quantity Rayleigh wishes to find the relationship with these variables is the quotient P of the pressure difference between two points along the tube and the distance between these points. The dimensions of this quantity are :

$$\frac{\mathrm{MLT^{-2}}}{\mathrm{L^2L}} = \mathrm{ML^{-2}T^{-2}} \ . \qquad \textbf{(196)}$$

And it's a matter of determining the exponents in the following relationship:

$$P \propto c^x v^y \rho^z w^n \ , \qquad \textbf{(197)}$$

knowing that the variables have dimensions :

c : L
v : $\mathrm{L^2T^{-1}}$ [33]
ρ : $\mathrm{ML^{-3}}$
w : $\mathrm{LT^{-1}}$

[29] Rayleigh 1892: 59-71.
[30] Reynolds 1883: 935-982.
[31] Darrigol 2005: 257.
[32] Rayleigh 1892: 59.
[33] Rayleigh gives this without further explanation. Since ρ has dimensions $\mathrm{ML^{-3}}$, this implies that μ has dimensions . $\mathrm{ML^{-1}T^{-1}}$

Homogeneity requires

$$x = n - 3, \qquad y = 2 - n, \qquad z = 1, \tag{198}$$

which results in the shape

$$P \propto c^{-3} \nu^2 \rho \left(\frac{cw}{\nu} \right)^n . \tag{199}$$

This is as far as dimensional analysis goes. The value of *n* must be determined by experiment, and is equal to 1 in the laminar case and around 2 in the turbulent case. Thus, in the first case :

$$P \propto c^{-2} \nu \rho w \tag{200}$$

and in the second :

$$P \propto c^{-1} \rho w^2 , \tag{201}$$

so that "the second power of velocity and the independence of viscosity are...inseparably linked." [34]

Rayleigh's use of dimensional analysis in the context of hydrodynamics influenced important work on turbulence in the early 20th century. Thus, in 1921, Theodore von Kármán (1881-1963), at the suggestion of Ludwig Prandtl (1875-1953), used dimensional analysis to obtain the expression for the velocity of a fluid in a pipe as a function of the distance from the wall, the shear stress on the wall, the density and the kinematic viscosity of the fluid. A few years later, in 1925, Prandtl himself employed dimensional

[34] The second power of velocity and independence of viscosity are thus inseparably connected." Rayleigh 1892: 60.

considerations (albeit different from Rayleigh's) to obtain the form of shear stress in turbulent flow as a function of fluid density.[35]

7. 2 - The spread of Rayleigh' method

As early as 1878, demonstrations of a similar nature to Rayleigh's can be found by another scientist, the French mathematician Joseph Bertrand.[36] Bertrand refers to Fourier, from whom he clearly draws inspiration, but does not mention Rayleigh, whose *Theory of sound* had appeared a year earlier.

For Bertrand, as for Rayleigh, the notion of invariance of equations is essential. He introduces his work with these remarks:

> Units are arbitrary, but the dependence of the quantities they measure imposes certain necessary relationships on their variations, in Physics as well as in Geometry and Mechanics. If, for example, we choose a unit of length that is ten times smaller, we must make the unit of volume a thousand times smaller, and if we double the unit of force, we must at the same time double the unit of mass. The formulas remain invariant to such changes: that's where their homogeneity lies.[37]

Bertrand's reference to geometry recalls Fourier's comparison between dimensional analysis and the fundamental lemmas in this branch of mathematics. [38]

Bertrand deals with two different situations: the time needed for a given point on a cable to reach a certain potential, and the time needed for a sphere to reach a given temperature. We'll start with the second problem, as the first is very similar to those we'll be dealing with later.

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[35] Kármán 1921: 233-252; Prandtl 1925, also in Prandtl 1961: vol.2 , 714-718. See also Darrigol 2005: 296-298.
[36] Bertrand 1878: 916-920.
[37] Bertrand 1878: 916.
[38] See chapter 1, 39.

More precisely, the situation envisaged by Bertrand is as follows: consider a homogeneous sphere of initial temperature V_o, placed in a medium maintained at 0 degrees. Bertrand is interested in "the time required for the mean temperature of the sphere...to become V."[39]

He considers four fundamental quantities: length, time, force and temperature. Curiously, he doesn't represent their units with capital letters (L, T, etc.), but reasons in terms of conversion factors. This approach is probably due to the fact that he proceeds by analogy with similitude, for which he offered a detailed discussion, including numerous examples, in 1848.[40] He imagines changing the fundamental units in such a way that :

- the unit length becomes α times smaller,
- the unit of time becomes β times smaller,
- the unit of force becomes γ times smaller,
- the temperature unit becomes δ times smaller.

This approach is similar to that of Fourier, insofar as the latter already reasoned on the basis of conversion factors.[41] But unlike Fourier, who made no assumptions about the nature of heat, Bertrand reduces the unit of heat to the unit of mechanical work, which becomes $\gamma\alpha$ times smaller when the above-mentioned changes of units are made.

Bertrand assumes that the duration whose shape he wishes to determine depends on the following variables :

V: temperature of the sphere at the desired time
V_o: initial temperature of the sphere
k: internal conductivity
C: specific heat
D: density
h: external conductivity
R: sphere radius

[39] Bertrand 1878: 919.
[40] See chapter 8, 276-283.
[41] See chapter 1, 38 ff.

We therefore have : $T = F\,(V,\, V_o,\, k,\, C,\, D,\, h,\, R)$ **(202)**

Bertrand gives himself two relations concerning the specific variables:

> It's easy to see that the coefficients k, C, D, h must appear in the ratios $\frac{k}{CD}$ and $\frac{h}{k}$ only, and that two different, geometrically identical bodies, for which these ratios are the same, being placed in the same calorific conditions, will heat up and cool down according to the same law.[42]

We now have :

$$T = F\left(V, V_0, \frac{k}{CD}, \frac{h}{k}, R\right). \qquad \textbf{(203)}$$

For this equation to be homogeneous, it is necessary that

$$\beta T = F\left(\delta V,\, \delta V_0, \frac{k}{CD}\frac{\alpha^2}{\beta}, \frac{h}{k\,\alpha}, R\,\alpha\right) \qquad \textbf{(204)}$$

for any value of coefficients α, β, γ, δ.

Although Bertrand does not explain why the new values of k, h, C and D are $\frac{k\gamma}{\delta\beta}\frac{h\gamma}{\alpha\beta\delta}$, ,$\frac{C\alpha^2}{\beta^2\delta}\frac{D\beta^2\gamma}{\alpha^4}$, it is clear that this results from Fourier's definitions of these constants. Since , $\delta V\ \delta V_0$ are the only expressions that involve δ, the quantities V and $V_o$must be involved in the form of a quotient. Since β appears among the variables only through $\frac{k}{CD}\frac{\alpha^2}{\beta}$, the quotient $\frac{CD}{k}$ must be a factor in the expression we're looking for. The coefficientα is then to the power -2 as a result of the previous step. Taking $R\alpha$ squared eliminates it, giving :

$$T = \frac{CDR^2}{k} F\left(\frac{V_0}{V}, \frac{Rh}{k}\right). \qquad \textbf{(205)}$$

[42] Bertrand 1878: 920.

This is equivalent to taking , , $\alpha = 1/R$ $\delta = 1/V$ $\beta = k/CDR^2$. Then we have :

$$T = \frac{CDR^2}{k} F\left(1, \frac{V_0}{V}, 1, \frac{hR}{k}, 1\right) = \frac{CDR^2}{k} \Phi\left(\frac{V_0}{V}, \frac{Rh}{k}\right) \quad \textbf{(206)}$$

Although incomplete, since it still involves an unknown function, this result makes it easy to determine the relationship between T and some of the remaining variables, if we experimentally determine its relationship with others. For example, it's easy to determine the relationship between the duration T and the radius R of the sphere, if we experimentally determine the relationship between T and h. If it turns out that T is inversely proportional to h, it follows from the above equation that T is proportional to R.

It is interesting to repeat what Bertrand does in the form of the method described by Rayleigh, for comparison. If we give ourselves the same additional relations as Bertrand, we can start from the relation :

$$T = F\left(V, V_0, \frac{k}{CD}, \frac{h}{k}, R\right). \quad \textbf{(207)}$$

We therefore have to determine x, y, z, p and q so that :

$$T \propto V^x \; V_0^{\,y} \left(\frac{k}{CD}\right)^z \left(\frac{h}{k}\right)^p R^q. \quad \textbf{(208)}$$

The dimensions of the sizes are as follows:

$$[V] = [V_0] = \theta \quad \textbf{(209 a)}$$
$$[k] = \mathrm{FT^{-1}\theta^{-1}} \quad \textbf{(209 b)}$$
$$[C] = \mathrm{L^2T^{-2}\theta^{-1}} \quad \textbf{(209 c)}$$
$$[D] = \mathrm{T^2FL^{-4}} \quad \textbf{(209 d)}$$
$$[h] = \mathrm{FL^{-1}T^{-1}\theta^{-1}} \quad \textbf{(209 e)}$$
$$[R] = \mathrm{L} \quad \textbf{(209 f)}$$

So : $$\left[\frac{k}{CD}\right] = \mathrm{L^2T^{-1}} \quad \text{and} \quad \left[\frac{h}{k}\right] = \mathrm{L^{-1}} \quad \textbf{(210)}$$

We therefore have the following system of equations:

For L: , $2z-p+q=0$ **(211 a)**
For T: $z=-1$ **(211 b)**
For θ : $x+y=0$, **(211 c)**

which gives

$$T \propto \frac{CDR^2}{k}\left(\frac{V_0}{V}\right)^a\left(\frac{Rh}{k}\right)^b , \qquad \textbf{(212)}$$

in which the exponents *a* and *b* can take any value. This form is more restrictive than Bertrand's, since its monomial character is admitted from the outset. But the linear combination of such forms makes it possible to obtain a form as general as Bertrand's, insofar as Bertrand's unknown function *F* admits a Taylor expansion.

The other problem Bertrand deals with concerns "the propagation of electricity in an insulated telegraph wire considered as indefinite."[43] He assumes that one end of the cable in question is brought to potential V_0 by a battery, and is interested in the time required for a defined point on the cable to reach a potential of a given value *V*. The variables on which Bertrand thinks this time depends are :

V_0 potential at which the end of the wire is connected
V: given potential at a given point on the wire
l: distance between the end of the wire and the point in question
R: wire resistance per unit length
C: electrical capacity of wire per unit length[44]
E: "quantity of electricity...which, in a current of unit intensity, passes through a section of wire in the unit of time."

Bertrand's relationship is :

$$T = F\left(V, V_0, l, R, C, E\right) \qquad \textbf{(213)}$$

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[43] Bertrand 1878: 916.
[44] "Quantity of electricity which charges the unit of length when the potential is equal to unity," Bertrand 1878: 917.

It takes as a unit of potential "that of a metallic sphere, of unit radius, charged on its surface with an electric mass equal to unit."[45] So, in dimensional terms :

$$[V]=\frac{[q]}{\mathrm{L}} \tag{214}$$

Bertrand considers that when changing units, "the unit of electric mass will be divided by $\alpha\sqrt{\gamma}$," which he obtains from Coulomb's law without a dimensioned coefficient. Consequently :

$$[V]=\frac{\mathrm{L}\left[\sqrt{F}\right]}{\mathrm{L}}=\sqrt{F}\ , \tag{215}$$

so that the corresponding conversion factor is . $\sqrt{\gamma}$

As for resistance, Bertrand starts by determining the conversion factor for current intensity. He does this on the basis of Ampère's law, again without a dimensioned coefficient, which implies $[i]=\sqrt{F}$ for the current, so that the conversion factor is $\sqrt{\gamma}$. These choices on Bertrand's part are unusual in that he doesn't stick to a single system of units: he expresses electric charge in electrostatic units, but expresses current in electrodynamic units. In this case, the $q=it$ equation no longer applies.

Having $[i]=\sqrt{F}$, Ohm's law gives $V = R\ i$ so R has no dimension. Bertrand finally considers resistance per unit length, whose conversion factor is $\frac{1}{\alpha}$. The capacitance per unit length, $C=\frac{q}{VL}$, has no dimensions. As for E, it can be expressed as $E=\frac{i\,A}{q\,t}$, which gives

[45] Bertrand 1878: 917.

$$[E] = \frac{\lfloor\sqrt{F}\rfloor \mathrm{L}^2}{\mathrm{L}\lfloor\sqrt{F}\rfloor \mathrm{T}} = \frac{\mathrm{L}}{\mathrm{T}}. \tag{216}$$

Bertrand thus obtains the identity :

$$\beta T = F\left(V\sqrt{\gamma},\, V_0\sqrt{\gamma},\, l\alpha, \frac{R}{\alpha},\, C,\, E\frac{\alpha}{\beta} \right). \tag{217}$$

By reasoning analogous to the previous case, he derives :

$$T = \frac{l}{E}\omega\left(\frac{V_0}{V}, lR,\, C \right). \tag{218}$$

He doesn't stop there, and gives himself the additional information (at the end of a long, non-dimensional argument into the details of which we won't go into) that ω must be proportional to R. Consequently, it must be proportional to lR. Bertrand thus obtains :

$$T = \frac{l^2 R}{E} F\left(\frac{V_0}{V}, C \right), \tag{219}$$

from which he concludes that "the time required for the potential to acquire a given value is consequently, for the same wire, proportional to the square of its length."[46]

Bertrand's demonstrations call for several comments. By reducing heat to mechanical quantities, he joins the mechanistic tradition inaugurated by Gauss and taken up by Maxwell in particular, although he does not give mechanical dimensions for temperature. Some of his choices of dimensions for electrical quantities are heterodox. Unlike Rayleigh, he did not develop a genuine method based on general rules of elimination. But his approach does have the advantage of allowing the intervention of an unknown function in the final result. He uses dimensional considerations to deduce as much as he can about a given problem, but he doesn't consider himself blocked if

[46] Bertrand 1878: 919.

indeterminacies remain in his final result. This is probably why, unlike Rayleigh, he doesn't insist on the limits of this kind of reasoning: he doesn't necessarily expect a complete equation anyway.

Another use of Rayleigh's method occurred more than a decade later, in 1892, also in France, in a controversy between engineer Claude Clavenad (1853-?) and physicist Emmanuel Carvallo (1856-1945). This controversy reveals the difficulties of the method. Although Carvallo didn't quote Rayleigh, it's highly likely that he drew inspiration from him: his reasoning was similar, and he emphasized the same limitations of the method as Rayleigh.

The controversy begins with an article by Vaschy,[47] , to which Clavenad responds.[48] The problem at hand concerns the relationship between the velocity of current propagation in a telegraph line and the relevant quantities: γ (capacitance per unit length) and λ (self-induction per unit length).[49] Vaschy obtains the equation: $v = \frac{1}{\sqrt{\gamma \lambda}}$. Clavenad, on the other hand, thinks the correct equation is $v = \sqrt{\frac{\lambda}{\gamma}}$. The reasons for this are not clear at this stage, except that he considers λ to be dimensionless and assigns γ the dimensions $\mathrm{T}^2 / \mathrm{L}^2$ (not without contradicting himself to achieve the latter result).[50] Carvallo defends the equation proposed by Vaschy.

[47] Vaschy 1892c: 1416-1419.
[48] Clavenad 1892b: 470-472.
[49] Vaschy 1892c: 1418.
[50] He uses the relation γ=q/e.*m.f.* and assigns the dimensions [*e.m.f.*] $\mathrm{T}^2 / \mathrm{L}$ to the charge *q,* without justifying this choice beyond specifying that he uses a system in which the dimensions of the *e.m.f.* are taken as one of the fundamental units. Clavenad 1892b: 471.

Carvallo believes that Rayleigh's method provides an equation, not just an *a posteriori* plausibility check. As long as we can decide experimentally which variables are relevant, the method is of great interest:

> This is a postulate [the dependence of v on λ, γ, the resistance per unit length ρ and the electromotive force employed E] but it is conceivable that, in certain cases, such a postulate can be established by experiment without, however, the form of the relationship being easily derived from experimental determinations.
> $$v = f\left(\gamma, \lambda, \rho, E\right)$$
> It is therefore worth discovering this shape through analysis. [51]

Carvallo describes this approach as "a true, logical and uniform research method." [52]

In his first reply to Clavenad, Carvallo gives no less than three demonstrations, the last two being special cases of the first. First, he uses the fundamental quantities L, M, T and e, where e represents the quantity of electricity (charge). He then makes the same calculations in the L, M, T system, replacing e with the dimensions of the charge as a function of these quantities, first according to the electrostatic system, then according to the electromagnetic system.[53] In all three cases, Carvallo takes as variables the capacitance per unit length γ, the self-induction coefficient per unit length λ, the resistance per unit length ρ, and the electromotive force E: [54]

$$v = f\left(\gamma, \lambda, \rho, E\right). \qquad \textbf{(220)}$$

It therefore has four variables, for a system initially with four fundamental quantities. Following the same reasoning as Rayleigh, Carvallo posits :

$$v = \gamma^{n} \lambda^{p} \rho^{q} E^{r} \quad . \qquad \textbf{(221)}$$

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[51] Carvallo 1892a: 165.
[52] Carvallo 1892a: 165.
[53] In the article that gave rise to the debate, Vaschy uses four fundamental quantities as well, one of which is electrical; nevertheless, he takes the capacity and not the quantity of electricity. Why Carvallo makes a different choice is unclear. As we shall see, the latter is not Clavenad's either.
[54] Carvallo 1892a: 165.

Since this equation must be homogeneous, $v^{-1}\gamma^{n}\lambda^{p}\rho^{q}E^{r}$ must not have any dimensions.

The five variables have the following dimensions:

$$[v] = \mathrm{LT}^{-1} \tag{222 a}$$
$$[\gamma] = \frac{C}{[\mathrm{L}]} = \mathrm{L}^{-3}\mathrm{T}^{-2}\mathrm{M}^{-1}\mathrm{e}^{2} \tag{222 b}$$
$$[\lambda] = \left[\frac{\text{self - induction}}{\text{longueur}}\right] = \frac{[p]}{[C]\mathrm{L}} = \mathrm{L\,M\,e}^{-2} \tag{222 c}$$
$$[\rho] = \left[\frac{\text{resistance}}{\text{longueur}}\right] = \frac{[E]}{[C]\mathrm{L}} = \mathrm{LT}^{-1}\mathrm{M\,e}^{-2} \tag{222 d}$$
$$[E] = \mathrm{L}^{2}\mathrm{T}^{-2}\mathrm{Me}^{-1} \tag{222 e}$$

The result is $v = \dfrac{\text{cons tan te}}{\sqrt{\gamma\lambda}}$, which is exactly the result proposed by Vaschy.

Carvallo then examines Clavenad's formula, substituting quantities by their dimensions in $v = \text{constante } \times\sqrt{\dfrac{\lambda}{\gamma}}$:

$$\mathrm{LT}^{-1} = \left(\mathrm{LMe}^{-2}\right)^{1/2}\left(\mathrm{L}^{-3}\mathrm{T}^{2}\mathrm{M}^{-1}\mathrm{e}^{2}\right)^{-1/2} \tag{223 }$$
$$\mathrm{LT}^{-1} = \mathrm{L}^{2}\mathrm{MT}^{-1}\mathrm{e}^{-2} \tag{224 }$$

This relationship imposes $\mathrm{e} = \mathrm{L}^{1/2}\mathrm{M}^{1/2}$, i.e. the charge must have the dimensions given to it in the electromagnetic system. Carvallo deduced that Clavenad's work corresponds to this special case, and this is what led him to study this system and its electrostatic counterpart.

To do this, he repeats the dimensional reasoning, replacing e with the appropriate dimensions. In the electrostatic system, the result is :

$$v = \frac{1}{\sqrt{\lambda}}\varphi(\gamma) \tag{ 225}$$

In the electromagnetic system, with $\mathrm{e} = \mathrm{L}^{1/2}\mathrm{M}^{1/2}$, the result is

$$v = \frac{1}{\sqrt{\gamma}} \psi(\lambda). \quad \textbf{(226)}$$

The formula obtained by Clavenad, $v = \text{constante} \times \sqrt{\frac{\lambda}{\gamma}}$, is therefore a special case of the latter result. Nevertheless, it is clear that Carvallo advocates the use of the first system, L, M, T, e :

> These remarks show the advantage of using the general system with four fundamental units... It provides one more condition equation than the others. It therefore makes it possible to fully determine the law sought, whereas the use of a particular system with three fundamental quantities does not make it possible to fully determine it, but leaves an arbitrary function.[55]

The only difference between the two scientists is that Carvallo uses four fundamental quantities (L, M, T, e) instead of Rayleigh's three (L, M, T).

Clavenad's reply has a similar structure in that he offers two demonstrations, the first representing his own opinion, and the second intended as a response to Carvallo. In the first, he has three fundamental quantities: L, T and E.[56] And he takes the capacitance (*C*) and self-induction of the line (L_i) rather than these quantities per unit length (γ and λ). Neither resistance nor resistivity are involved. The reason is that he believes this variable is not independent: indeed he believes that once capacitance is given, resistance "results as a function of time." As proof of this, he cites the following equations, which he seems to understand as dimensional:

$$C = \frac{Q}{E} = \frac{I\,\mathrm{T}}{E} \ , \quad \textbf{(227)}$$

and since $I = \frac{E}{R}$:

$$C = \frac{\mathrm{T}}{R}. \quad \textbf{(228)}$$

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[55] Carvallo 1892a: 168.

[56] We use E without italics to designate the electromotive force as a fundamental quantity, and *E* italized for the value of the electromotive force.

The decision not to take resistance as a variable is what allows Clavenad to achieve a complete result using the method in question with his system of three fundamental quantities. The result of the homogeneity constraint is then

$$v = \text{constante} \times \sqrt{\frac{L_i}{C}} \qquad \textbf{(229)}$$

in accordance with its previous result. [57]

Clavenad then sets out to demonstrate that Carvallo's work is wrong. His reasoning is based on the use of the quantities C and L_i rather than γ and λ. It therefore depends on the choice of variables,[58] but Clavenad believes that there is nothing arbitrary about this. The reasons for this conviction are complex - not to say confusing. With this choice of variables, Clavenad ends up with an insoluble system of equations. Instead of questioning his premises, he pursues a highly questionable line of reasoning. After some trial and error, he takes the dimensional exponents of resistance and electromotive force, both equal to 0. This gives him a system of four equations with two unknowns. He then adds together three of these equations (those corresponding to L, M and e). With the equation for T, he thus has a system of two equations, which finally gives him the result he's looking for: ½ for the exponent of λ and $-\frac{1}{2}$ for that of $.\gamma$

Carvallo is quick to criticize Clavenad for the unorthodox way in which he "solves" his equation system: [59]

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[57] Clavenad 1892a: 216. Recall that he used to obtain $v = A\sqrt{\frac{\lambda}{\gamma}}$, but due to his change of notation, he now represents λ by L_i and γ by C.

[58] At this stage of the debate, the question also seems to hinge on a disagreement over the dimensions to be assigned to quantities. However, this disagreement is deeply rooted in Clavenad's conceptions of the quantities themselves.

[59] Carvallo 1892a: 319-321.

> If the four equations he obtains to determine the four unknowns n, p, q, r are incompatible, they will be just as incompatible by assigning a zero value a priori to one or more of the unknowns.[60]

He offers an alternative demonstration to Clavenad's to show that it actually leads to his own result (and thus to Vaschy's). Interestingly, he agrees to leave out the variables ρ and E. This is certainly due to the fact that these quantities do not appear in either of the two equations involved ($v = \frac{\text{constante}}{\sqrt{\gamma\,\lambda}}$ and $v = \text{constante} \times \sqrt{\frac{L_{\mathrm{i}}}{C}}$). Carvallo also agrees to take C and L_i as variables in the expression of v rather than γ and λ[61], but insists that in this case these variables are not sufficient, and that the length ℓ of the line must be added. Carvallo thus has :

$$v = f\left(C, L_{\mathrm{i}}, \ell\right). \tag{230}$$

The expression that must be dimensionless is therefore : $v^{-1}C^{n}L_{\mathrm{i}}{}^{p}\ell^{r}$ and given that these quantities have dimensions :

$$[v] = \mathrm{LT}^{-1}, \qquad [C] = \mathrm{L}^{-2}\mathrm{T}^{2}\mathrm{M}^{-1}\mathrm{e}^{2}, \qquad [L_{\mathrm{i}}] = \mathrm{L}^{2}\mathrm{Me}^{-2}, \qquad [\ell] = \mathrm{L}, \tag{231}$$

we must have

$$v = \text{constante} \times \sqrt{\frac{\ell}{CL_{\mathrm{i}}}}. \tag{232}$$

Dividing the numerator and denominator by ℓ , this relationship can be rewritten :

$$v = \frac{\text{constante}}{\sqrt{\gamma\,\lambda}}, \tag{233}$$

in accordance with Vaschy's result.[62]

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[60] Carvallo 1892a: 320.

[61] Carvallo 1892a: 320. In fact, Carvallo now refers to them as q and s respectively, but for the sake of clarity we'll follow the notation introduced by Clavenad.

[62] Carvallo 1892a: 321.

Clavenad was not convinced, however, and responded one last time to Carvallo's work.[63] As might be expected, he directs his criticism precisely at the point Carvallo uses to modify Clavenad's earlier demonstration: he finds it unacceptable to express propagation velocity as a function of line length. He notes that the relation $v = f(C, L_i, \boldsymbol{\ell})$ on which Carvallo's latest demonstration is based can be written :

$$\frac{\ell}{t} = f(C, L_i, \ell) \tag{234}$$

where t is the time taken for the signal to propagate from one end of the line to the other. Clavenad's reasoning then becomes particularly surprising:

> The length of the line is therefore, for some reason, both in the first term of ([56]) and in the second .[64]

And he replaces the above equation with another relationship, in which he isolates ℓ , and expresses it in terms of the other three quantities: $\ell = F(C, L_i, t)$. He then applies the method to $\ell = F(C, L_i, t)$. In this case, the expression $\ell^{-1} C^n L_i^{\,p} t^r$ should be dimensionless, but he makes the mistake of taking 1 and not -1 for the exponent of ℓ . He then obtains a system of 4 equations with 3 unknowns that is insoluble, and deduces that Carvallo's reasoning is wrong. Again, it's only by adding 3 of the equations together, and taking $r = -1$ after some dubious reasoning, that Clavenad arrives at the result he defends.[65]

[63] Clavenad 1892c: 661-666.
[64] Clavenad 1892c: 662.
[65] Clavenad also offers what appears to be an alternative demonstration. Rather than replacing $\frac{\ell}{\boldsymbol{t}} = f(C, L_i, \ell)$ by $\ell = F(C, L_i, t)$, he replaces it by $t = F(C, L_i, \ell)$. He arrives at the desired result through reasoning similar to that just described (including assigning the wrong dimension exponent, to T this time).

Carvallo's only response to Clavenad's work was a short article in which he reiterated his positions, and showed that the equation he had used, $\frac{\ell}{t} = f(C, L_i, \ell)$ by $\ell = F(C, L_i, t)$, could not be replaced. [66]

While it's clear that Clavenad's reasoning often leaves something to be desired, the controversy itself is no less revealing. The plurality of demonstrations stems from the freedom allowed by the method itself, which encourages misunderstandings between the two protagonists. This freedom takes two different forms. Firstly, different fundamental units can be used. Certainly, since the final result concerns the relationship between a certain quantity and other variables, the choice of dimensional system should not affect it. However, this choice leads to different results when *partial results* are involved. Carvallo presents such cases in his first article. Secondly, as Rayleigh pointed out, it is necessary to choose the variables. This is the main stumbling block between Clavenad and Carvallo. Of course, it's perfectly possible to use the method with different groups of variables: the result is different, but not necessarily incompatible, equations. Here, however, there is a fundamental disagreement. Clavenad insists on using the *absolute* capacity and self-induction coefficient of the line. Carvallo, on the other hand, believes that these quantities should be taken per unit length, or else that the length of the line should be added to the number of variables. The underlying reasons seem to lie in a difference of opinion about the physical phenomena themselves. The controversy is therefore symptomatic of the limits

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[66] In particular, $v = constante\sqrt{\frac{\ell}{CL_i}}$, which corresponds to the first relationship, cannot be represented by the second: $\frac{\ell}{t} = constante\sqrt{\frac{\ell}{CL_i}}$ implies $\frac{1}{t} = constante\sqrt{\frac{1}{CL_i}}$, in which case the line length disappears entirely from the relationship.

of the “method of dimensions”. It also reveals the difficulty some people have in understanding it, even if the liberties Clavenad allows himself in manipulating systems of equations seem to be enough to discredit him.

In short, Rayleigh's "“method of dimensions”" can be seen as the first development of dimensional analysis as a tool for discovery. Although it bears a strong resemblance to the method used to check the homogeneity of equations, this resemblance only becomes apparent at a later stage: in Rayleigh's early writings of 1871, we find more a series of trial-and-error reasonings than a method as such. It was in 1877 with *Theory of sound* that this formalization took place, and Rayleigh also applied dimensional analysis to similarity problems. By 1871, however, it was clear that the type of reasoning followed by Rayleigh could lead to the discovery of new equations and even decide between different models.

As Rayleigh explains in his 1877 treatise, the dimensional method has its limitations. Firstly, a complete result (not involving an unknown function) can only be obtained from dimensional considerations alone when the number of variables involved is at most equal to that of the fundamental quantities (generally three). It is still possible to obtain a complete result by taking into account already known relationships between some of the variables, or to settle for a partial result, involving an unknown function of some of the variables, but restricting the form of the law sought - this is Bertrand's approach.

Secondly, the method requires knowledge of all the variables on which the quantity in question is likely to depend. This is far more problematic, since neglecting some of the variables will lead to erroneous results. The difficulty of determining these variables is amply illustrated by the controversy between Carvallo and Clavenad.

Thirdly, Rayleigh's method requires the choice of a given set of fundamental quantities. While this freedom is legitimate and cannot in itself lead to inaccurate results, it can lead to confusion - indeed, it facilitates misunderstandings in this very controversy between Carvallo and Clavenad. Last but not least, the "“method of dimensions”" naturally requires us to determine the dimensions of the different variables. However, in a context where electrical and magnetic quantities are assigned different dimensions depending on the system used - or even within the same system - this step can be problematic.

It's interesting to note that the earliest examples we find of the use of Rayleigh's method, or similar reasoning, are in France. Particularly curious is the fact that the first example, due to Bertrand and dating from 1878, bears a much closer resemblance to Rayleigh's approach of 1871, than to his method, which he expounded the year before, in 1877. Nevertheless, in the 1892 controversy between Carvallo and Clavenad, Rayleigh's method, or some perverted version of it, was used.

Chapter 8: Similarity

8. 1 - Introduction

The development of dimensional analysis from the 1870s onwards was accompanied by considerations of another issue: similarity. In particular, Rayleigh sought in the similarity of physical systems the reasons for the power of his "dimensional method." Similarity concerns the comparison of the behavior of identical systems within a change of scale. We can distinguish between simple geometric similarity, in which we are only interested in changes in geometric magnitudes (cartography, for example), and dynamic similarity. The latter, which is what we're really interested in here, studies how a change in magnitude of one of a system's characteristics (usually its size) affects its dynamic behavior (the forces to which it is subjected, for example). The aim is therefore to compare certain physical quantities in systems of different scales. Reflections on this subject can be found well before the 19th century - in fact, as early as the 16th century, as we shall see.

8. 2 - Similarity before the 19th century

Galileo's considerations of similarity can already be found in the context of a specific problem,[1] that of the "resistance" of objects to a force tending to deform (or break) them, i.e. their resistance to bending. Galileo was more interested in the effect of object shape than size. This nevertheless led him to establish relationships in which their size plays a part. Dimensional considerations are implicit in his reasoning.

Galileo compares the bending strength of a solid cylinder with that of a hollow cylinder (i.e. a tube). He considers cylinders of the same length and made of the same material, but such that the outside and inside diameters of the hollow cylinder are generally different from the diameter of the solid cylinder.

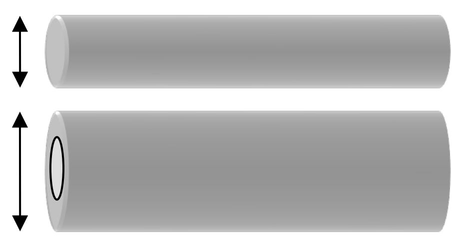

Figure 8.1

In an initial demonstration of little interest to us, Galileo concludes that for two cylinders whose cross-sections have the same area (not taking into account the hollow part of the tube), the quotient of the bending strengths of the two objects is equal to the quotient of their outside diameters. He then studies the general case where the two cylinders (one hollow, the other solid) have different transverse areas. In order to be able to compare them,

[1] Galileo 1638: Second day, [187]-[189].

he considers the intermediate case of a cylinder whose cross-sectional area is the same as that of the tube,[2] cylinder represented by object 2 in the diagram below. [3]

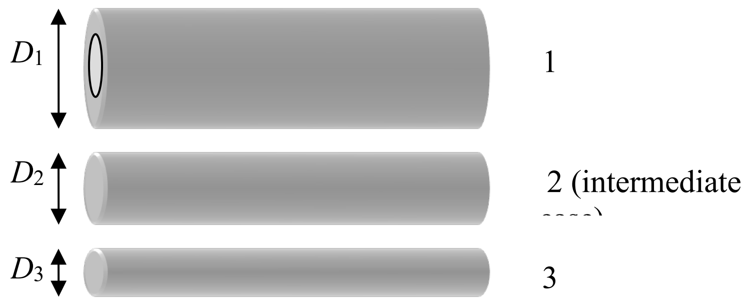

Figure 8.2

If we call R the bending strength of one of these objects, according to the previously demonstrated result we have :

$$\frac{R_1}{R_2} = \frac{D_1}{D_2} \qquad \textbf{(235)}$$

Galileo then defines a length V such that :

$$\frac{V}{D_3} = \frac{D_3}{D_2} \qquad \textbf{(236)}$$

He demonstrates that the quotient $\dfrac{R_1}{R_3}$ of the resistances is equal to the quotient $\dfrac{D_1}{V}$. This demonstration is based on the idea that the ratio of the flexural strengths of cylinders 2 and 3 is equal to the ratio of the squares of their diameters. [4]

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[2] It explains how to determine, by geometric construction, the diameter of a cutting cylinder with the same area as a given tube.

[3] For the sake of simplicity, we do not follow Galileo's notation.

[4] Galileo declares that the ratio of the resistances of the cylinders is equal to the ratio of the *cubes* of their diameters, but his demonstration does not lead to the above result, which he indicates twice. It does, however, if we take the ratio of squares.

Galileo doesn't justify this idea, but it does imply that he recognized the importance of the object's geometric dimensions.

After Galileo, a more convincing case of the use of similitude can be found in Newton's *Principia*. It's found in a section entitled "The Motion of Fluids, and the Resistance Undergone by Projectiles."[5] This is no longer a question of static similitude. Newton sets out to study the conditions under which two systems that are similar in the geometrical sense at a given moment will remain so thereafter: this is the question of dynamic similarity. He defines what he means by "similar" as follows:

> Similar bodies in similar situations are said to be moved in similar motions and proportional times, when their situations at the end of these times are always similar to each other; so suppose we compare the particles of one system with the corresponding particles of the other. The times will be proportional in which the similar and proportional parts of similar figures are described by the corresponding particles.[6]

The similar systems in Newton's mind can therefore be represented schematically as follows:

Indeed $\frac{V}{D_3} = \frac{D_3}{D_2}$ implies $V = \frac{D_3^{\ 2}}{D_2}$; cylinders 2 and 3 being full, it is clear that $\frac{R_2}{R_3} = \frac{D_2^{\ 2}}{D_3^{\ 2}}$; therefore :

$$\frac{R_1}{R_3} = \frac{R_1}{R_2} \cdot \frac{R_2}{R_3} = \frac{D_1}{D_2} \cdot \frac{D_2^{\ 2}}{D_3^{\ 2}} = D_1 \cdot \frac{D_2}{D_3^{\ 2}} = \frac{D_1}{V} \ .$$

[5] "The motion of fluids, and the resistance made to projected bodies"Newton 1687: section vii, 219.

[6] "Like bodies in like situations are said to be moved among themselves with like motions and in proportional times, when their situations at the end of those times are always found alike in respect of each other; as suppose we compare the particles in one system with the corresponding particles in the other. Hence the times will be proportional, in which similar and proportional parts of similar figures will be described by corresponding particles." Newton 1687: section vii, proposition 32, 219.

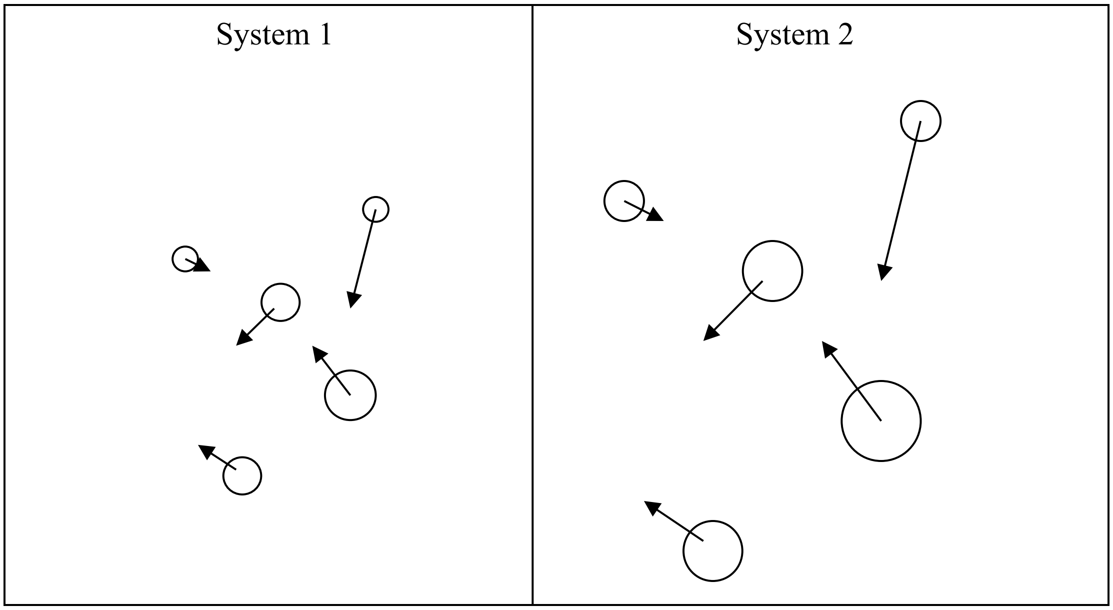

Figure 8.3

The number of particles is the same. The distances between particles and particle diameters are proportional, with the same proportionality coefficient. So particle densities in corresponding volumes are also proportional. The directions in which the particles move are also the same.

Newton offers the following theorem:

> Suppose two similar systems of bodies composed of an equal number of particles, and that the corresponding particles are similar and proportional, each in one system to each in the other, and that they are situated in the same manner between them, and have the same given ratio of density to each other; and suppose that they begin to move between each other in proportional times, and with similar motions (that is, those in one system relative to each other, and those in the other system relative to each other). *And if the particles in the same system do not touch each other, except at moments when they are reflected; nor do they attract or repel each other, except with accelerating forces inversely proportional to the diameters of the corresponding particles, and directly proportional to the squares of their velocities*: in this case, the particles in these systems will continue to move with each other in similar ways and in proportional times.[7]

[7] My italics. "Suppose two similar systems of bodies consisting of an equal number of particles, and let the corresponding particles be similar and proportional, each in one system to each in the other, and have a like situation among themselves, and the same given ratio of density to each other; and let them begin to move among themselves in proportional times, and with like motions (that is, those in one system among one another, and those in the other among one another). And if the particles that are in the same system do not touch one another, except in the moments of reflection; nor attract, nor repel each other, except with

Newton then explains that, because of his first law, the two systems will remain similar until the particles collide. What's more, since their motions are similar, the collisions within each system will also be similar, and consequently the particles will reflect in the same way. The two systems will therefore remain similar even after their particles collide.

The condition he imposes on forces when there are any only becomes clear when he explains the following theorem:

> If we assume the same things, then the parts of the systems undergo a resistance proportional to the square of the quotient of their velocities, to the square of the quotient of their diameters, and directly proportional to the quotient of the densities of these parts of the system.[8]

Newton associates these forces with centripetal forces, of the form :

$$F = \frac{mv^2}{r}$$

where r is the radius of curvature.

So the similar forces in the two systems must obey the relation :

$$\frac{F_2}{F_1} = \frac{\frac{m_2}{m_1}\frac{v_2^{\ 2}}{v_1^{\ 2}}}{\frac{r_2}{r_1}}$$

The mass ratio can be expressed as follows:

$$\frac{m_2}{m_1} = \frac{\rho_2 V_2}{\rho_1 V_1} = \frac{\rho_2 d_2^{\ 3}}{\rho_1 d_1^{\ 3}}$$

accelerative forces that are inversely as the diameters of the corresponding particles, and directly as the squares of the velocities: I say, that the particles of those systems will continue to move among themselves with like motions and in proportional times." Newton 1687: section vii, proposition 32, theorem 26, 219.

[8] "The same things being supposed, I say, that the greater parts of the systems are resisted in a ratio compounded of the squared ratio of their velocities, and the squared ratio of their diameters, and the simple ratio of the density of the parts of the systems." Newton 1687: section vii, proposition 33, theorem 27, 220.

where d_1 is the diameter of a particle in system 1 and d_2 that of the equivalent particle in system 2, V_1 is a given volume in system 1 and V_2 the similar volume in system 2, and ρ_1 and ρ_2 are the corresponding densities. Furthermore, the geometric similarity of the two systems implies the proportionality of all distances; therefore :

$$\frac{d_2}{d_1}=\frac{r_2}{r_1} \text{ and } \frac{m_2}{m_1}=\frac{\rho_2 r_2^{\,3}}{\rho_1 r_1^{\,3}}$$

This leads to :

$$\frac{F_2}{F_1}=\frac{\rho_2}{\rho_1}\frac{r_2^{\,2}}{r_1^{\,2}}\frac{v_2^{\,2}}{v_1^{\,2}}.$$ [9]

Newton then applied his theorem to the case of solid objects moving in fluids:

> Corollary 1: Therefore if these two systems are two elastic fluids, such as our air, and their parts are at rest relative to each other; and two similar bodies, of magnitude and density proportional to the parts of these fluids, and similarly situated among them, find themselves projected in any way along similar directions; and the accelerating forces with which the particles of the fluids act upon each other are inversely proportional to the diameters of the projected bodies and directly proportional to the square of their velocities, these bodies will have similar motions in the fluids, in proportional times, and will travel similar distances and proportional to their diameters.
>
> Corollary 2: Consequently, a body projected into such a fluid and moving rapidly encounters a resistance that is almost as great as the square of its velocity. For if the forces with which particles at a distance from each other interact were to increase as the square of the velocity, the projected body would encounter a resistance also going as the square of the velocity, exactly; and so in a medium whose separate parts interact with no forces, the resistance goes as the square of the velocity, exactly.[10]

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[9] Newton 1687: section vii, proposition 33, 220-221. He is referring to the passage where he states that centripetal forces are like the square of velocities and the inverse of radii. Newton 1687: book I, section ii, proposition 4, corollary 1, 36. See also Wright 1983: 188-191.

[10] Newton 1687: section vii, proposition 33, 221.

Cor.I. Therefore if those systems are two elastic fluids, like our air, and their parts are at rest among themselves; and two similar bodies proportional in magnitude and density to the parts of the fluids, and similarly situated among those parts, be in any manner projected in the direction of lines similarly posited; and the accelerative forces with which the particles of the fluids act upon each other are inversely as the diameters of the bodies projected and directly as the squares of their velocities; those bodies will excite similar motions in the fluids in proportional times, and will describe similar spaces and proportional to their diameters.

Newton thus deduced from his theorem that objects moving in fluids encounter forces proportional to the square of their velocities.

8. 3 - Similarity in the mid-19th century

The scientist who first published a discussion of similarity in the period we're interested in was Bertrand, as early as 1848.[11] His work is explicitly based on what he calls "Newton's theorem. He set out to demonstrate it, and then to present its applications:

> This theorem constitutes a veritable theory of similarity in mechanics; we can see that, given any given system, there is an infinite number of possible systems that can be regarded as similar to this one, and that instead of a single relationship of similarity as in geometry, there are four to consider; namely, that of lengths, that of times, that of forces and that of masses: one of these relationships is, according to Newton's theorem, a consequence of the other three.[12]

In addition to the fact that Newton didn't explicitly refer to mass ratios, Bertrand interprets his theorem more broadly than Newton intended: he includes any mechanical system, whereas Newton limited himself to particular fluid models.

What Bertrand means by Newton's theorem is the following. He considers two similar systems, by which he means two systems characterized by the same physical quantities and capable of similar motions, but of different scales (with one exception, all

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Cor.II. Therefore in the same fluid a projected body that moves swiftly meets with a resistance that is as the square of its velocity, nearly. For if the forces with which distant particles act upon one another should be augmented as the square of the velocity, the projected body would be resisted in the same squared ratio accurately; and therefore in a medium, whose parts when at a distance do not act with any force on one another, the resistance is as the square of the velocity, accurately.

[11] Bertrand 1848: 189 - 197.

[12] Bertrand 1848: 190.

the applications he discusses below concern systems of different sizes). His reasoning is based on the quotients of the quantities of the two systems, which he designates :

α Quotient of the lengths of the two systems.
β Quotient of the masses of the homologous points of the two systems.
γ Quotient of forces of homologous points
ε Quotient of times for systems to be in the same state (defined according to their characteristics; for example, times for two pendulums to describe the same angle).

The theorem then states that for the two systems to be similar, these quotients must have the relationship :

$$\varepsilon^2 = \frac{\alpha\beta}{\gamma}. \tag{237}$$

To demonstrate the theorem, Bertrand uses d'Alembert's principle and the principle of virtual work, according to which the motion of the system must satisfy the relation :

$$\sum\left(X - m\frac{d^2x}{d^2t}\right)\delta x + \left(Y - m\frac{d^2y}{d^2t}\right)\delta y + \left(Z - m\frac{d^2z}{d^2t}\right)\delta z = 0\,, \tag{238}$$

where *X*, *Y* and *Z* are the components of the forces applied to the various material points of the system, *m* their masses, *x*, *y* and z their spatial coordinates as a function of time, δx, δy and δz any virtual displacements of these points compatible with the links of the system.

If one of the two systems is characterized by points with positions *x*, *y* and *z*, and masses *m*, moving in durations *t* under the action of force components *X*, *Y* and *Z*, the points of the second system must have positions α x, α y and α z, masses β m and move in durations ε t under the action of γ X, γ Y and γ Z. Substituting these new values into the above equation (as well as $\alpha\delta$ x, $\alpha\delta$ y and $\alpha\delta$ z), the new motion satisfies d'Alembert's conditions, provided that

$$\left(\gamma - \beta\frac{\alpha}{\varepsilon^2}\right)\alpha = 0\,. \tag{239}$$

The result is: $$\varepsilon^2 = \frac{\alpha\beta}{\gamma}. \qquad \textbf{(240)}$$

As the form of this equation suggests, most of Bertrand's applications involve establishing the ratio of durations for two similar systems, in terms of the relationships between the parameters of these two systems. The first case he deals with is that of two equal masses M and M', both initially at rest, but located at different distances from a center of force O. If they are attracted to O by a force proportional to their distance from this point, both masses will reach O at the same moment. In this case, $\beta = 1$ and $\gamma = \alpha$. We therefore have $\varepsilon = 1$, i.e. . $\frac{t'}{t} = 1$ [13]

More generally, if the two masses are attracted by a force proportional to the power *n* of the distance to O, the ratio of the times for them to reach O is equal to the ratio of the initial distances to the power : $\frac{1-n}{2}$

$$\frac{t'}{t} = \left(\frac{\mathrm{OM'}}{\mathrm{OM}}\right)^{\frac{1-n}{2}}. \qquad \textbf{(241)}$$

Indeed $\beta = 1$, but this time $\gamma = \alpha^n$, so . $\varepsilon^2 = \frac{\alpha}{\alpha^n}$ [14]

Bertrand then considers several types of vibration. The simplest case consists of two pendulums of different lengths l and l', subjected to different gravitational accelerations, g and g'. He demonstrates that "the ratio of the times required to traverse similar arcs" is :

[13] Bertrand 1848: 192.

[14] Here $\frac{OM'}{OM} = \alpha$. Bertrand 1848: 192.

$$\frac{t}{t'} = \frac{\sqrt{\frac{l}{g}}}{\sqrt{\frac{l'}{g'}}}. \qquad \textbf{(242)}$$

According to Newton's second law, the quotient of accelerations $\frac{g}{g'}$ must be $\frac{\gamma}{\beta}$. This gives

$\varepsilon^2 = \alpha\frac{g'}{g}$, where . $\alpha = \frac{l}{l'}$ [15]

The second type of vibration involves a string of non-negligible mass of length *l*, "tensioned by a weight *P*"[16] - a case which differs from the previous one in that the mass of the string is no longer assumed to be negligible. The total mass of the system is *m*.[17] Bertrand shows that if we consider two such systems, the ratio of durations will be :

$$\frac{t}{t'} = \frac{\sqrt{\frac{ml}{P}}}{\sqrt{\frac{m'l'}{P'}}}. \qquad \textbf{(243)}$$

The third and final case of vibrations considered consists of two strings stretched by the same weight but of different lengths, and to which are attached "any number of similarly placed cursors." Bertrand shows that, provided the masses of the cursors are in the same ratio as the length of the strings, the ratio of the times for all oscillations is equal to the ratio of the corresponding lengths of the two systems: here $\gamma = 1$ and $\beta = \alpha$, so . $\varepsilon^2 = \alpha^2$ [18]

[15] Bertrand 1848: 193.
[16] Bertrand 1848: 193.
[17] The weight P is apparently exerted at the end of the rope, being due to the mass of the pendulum weight, but this is not entirely clear.
[18] Bertrand 1848: 193-194.

Bertrand also considers a number of situations for which he obtains ratios other than time ratios. For example, he takes two masses m and m' that travel in circles of different radii (r and r'), with the same speed. Their periods t and t' are therefore different. Bertrand demonstrates that in this case, the centripetal forces must have the ratio :

$$\frac{f}{f'} = \frac{\frac{mr}{t^2}}{\frac{m'r'}{t'^2}} \quad . \tag{244}$$

This relationship immediately follows from . $\alpha = \frac{r}{r'}$ [19]

Bertrand also considers the more complex case of machines: he demonstrates that for two similar turbines, the ratio of the power they consume is equal to the ratio of the power they deliver, if the ratio of their angular velocities is equal to the inverse ratio of the square roots of their linear dimensions. The ratio of the weights of objects that turbines can lift, [20] β, is α^3 (equal to the ratio of their volumes, since they are assumed to be made of the same material). Since the forces involved are the weights of these objects, their ratio γ is also α^3 (proportional to the corresponding masses). The formula therefore implies a ratio of durations :

$$\varepsilon = \sqrt{\frac{\alpha\alpha^3}{\alpha^3}} = \sqrt{\alpha}\,, \tag{245}$$

[19] Bertrand 1848: 192-193.

[20] "Resistances, which can always be viewed as weights to be lifted with uniform speeds." Bertrand 1848: 195.

which corresponds to turbines with angular velocities in the ratio $.\frac{1}{\sqrt{\alpha}}$[21] Consequently, the ratio of power supplied by the turbines, which is of the form $\frac{\gamma\,\alpha}{\varepsilon}$ (ratio of forces multiplied by ratio of heights to which the turbines raise the objects, divided by the ratio of durations), gives :

$$\frac{\alpha^3\,\alpha}{\sqrt{\alpha}} = \alpha^3\sqrt{\alpha}\,. \tag{246}$$

To obtain the ratio of power consumed by the turbines, Bertrand considers the variation in potential energy of the water as it falls. The ratio of water masses β is α^3 , the gravitational acceleration is obviously the same in both cases, so the corresponding ratio is trivially 1, the ratio of water displacements is α, and the ratio of durations is always $\sqrt{\alpha}$. So the ratio of power expended is also :

$$\frac{\alpha^3\,\alpha}{\sqrt{\alpha}} = \alpha^3\sqrt{\alpha}\,. \tag{247}$$ [22]

Finally, Bertrand considers a situation that is not strictly speaking a change of scale, but a change of other parameters of the systems under consideration. Thus, he derives the ratio of propagation velocities v and v' of a wave in gases of different densities *d and d'*, and elasticities e and e'. For the ratio of these velocities, he obtains :

$$\frac{v}{v'} = \frac{\sqrt{\frac{e}{d}}}{\sqrt{\frac{e'}{d'}}}\,. \tag{248}$$

[21] Indeed $\omega = \frac{v}{r}$ implies that the angular velocity ratio is $.\frac{\alpha}{\varepsilon}.\frac{1}{\alpha}$

[22] Bertrand 1848: 195-196.

Indeed, the ratio of speeds is the quotient of the ratio of displacements by the ratio of durations, . $\frac{\alpha}{\varepsilon}$ α is equal to 1 here, so $\frac{v}{v'} = \frac{1}{\varepsilon}$ The ratio of forces, γ, is $\frac{e}{e'}$ and that of masses, β, is equal to the ratio of densities (since $\alpha = 1$), $\frac{d}{d'}$. Bertrand's condition implies : $\frac{1}{\varepsilon^2} = \frac{\gamma}{\alpha\beta} = \frac{\gamma}{\beta}$, hence the above result.[23]

Bertrand's main interest in this method is its usefulness to the engineer:

> We can see...how far-reaching is the principle to which I have tried to draw attention, and how diverse are the applications that can be made of it.... There can be...great utility in determining, in certain cases, the relationships that exist between the motions of two systems, even when each of them would not be susceptible to rigorous theoretical determination. This principle should be applied, for example, whenever we are trying to predict, by means of small-scale experiments, the value of a mechanical invention whose complete realization would be too costly.[24]

The theoretical interest he sees in it is less, and he even insists on the limits of the method:

> It is true that only relationships can be deduced from it, and that it can therefore only be used to solve a question once a similar one of equivalent analytical difficulty has already been solved. [25]

Bertrand also mentions to whom we owe the demonstration of the various illustrations he gives when they are not obvious.

Further considerations of similarity can be found in the writings of a naval engineer, Ferdinand Reech (1805-1884), long-time director of the Ecole Spéciale d'Application du Génie Maritime in Lorient. When he speaks of similarity, Reech has ship models in mind. His work is in line with the development of theoretical studies on navigation, which took

[23] Bertrand 1848: 194-195.
[24] Bertrand 1848: 196.
[25] Bertrand 1848: 196.

off in the first half of th e 19th century with the development of steam navigation. In particular, there was the problem of the resistance of these vessels, which were considerably faster than their predecessors.[26] Reech's discussion is nonetheless general. He had certainly obtained his derivations before Bertrand, as he refers to them in a memoir of 1844, but did not publish the details until 1852.[27]

Like Bertrand, Reech establishes relationships between the ratios of homologous quantities in similar systems (quantities he designates by lower case letters) and refers to Newton. He is interested in systems of different sizes, and in particular in the relations between the ratios of forces on these systems, as a function of the ratios of their lengths, and also as a function of the ratios of their densities and the ratios of their velocities. He distinguishes between the statics of rigid bodies, the statics of flexible bodies, and dynamics.

The statics of rigid bodies is the simplest case: "It is sufficient for a system of forces P, P', P" to be in equilibrium on a certain figure for all the same forces, multiplied by any number *f*, to be in equilibrium on the same figure and on any figure similar to it.[28] In this case, the ratio of forces, *f*, is independent of that of lengths, *l*. Reech believes, however, that this is true only in the "ideal case" where systems are composed of bodies "with finite volumes at great distances from each other," and "as long as the volumes of the various bodies do not [exceed] the limits beyond which these volumes would penetrate each other."[29] On the other hand, in the "real case" where these conditions do not apply, the ratios f and l are no longer independent, but obey the relationship $f = dl^3$ where d

[26] Darrigol 2005: chapter 7.
[27] Reech 1844: 166, Reech 1852: 265-275.
[28] Reech 1852: 265.
[29] Reech 1852: 265-266.

represents the ratio of densities. Reech's view is that, in this case, we are obliged to include body weights among the forces. These weights obey the above-mentioned relationship (since they are proportional to masses), and all homologous forces between two systems must have the same ratio *f*. It follows that all forces must obey this relationship.[30]

In the case of the statics of flexible bodies, the application of similarity is more delicate, since the forces responsible for the deformation of one system do not all bear the same relationship to their counterparts in the other system. Reech nevertheless gives a valid relationship for a specific case: if we consider two initially undeformed systems and then subject them to deforming forces, "invoking the ordinary formulas of the theory of strength of materials...for the same species of material, and for the same elastic quality," we will have between the ratio f of homologous forces and the ratio l of lengths the relationship $f = l^2$. Although he does not specify it, it seems clear that Reech means by l the ratio of lengths that are not deformed by the force in question, corresponding to the homologous surfaces on which the homologous forces apply. [31]

The most interesting case is that of dynamics. Reech distinguishes between two situations, as he did for statics: one where weights are of no concern, and one where they must be taken into account. In the first case, Reech begins by separating the tangential and normal components of a point's acceleration:

$$\frac{d^2x}{dt^2} = \frac{v\,dv}{ds}\cos\alpha + \frac{v^2}{r}\cos a \qquad \textbf{(249 a)}$$

$$\frac{d^2y}{dt^2} = \frac{v\,dv}{ds}\cos\beta + \frac{v^2}{r}\cos b \qquad \textbf{(249 b)}$$

$$\frac{d^2z}{dt^2} = \frac{v\,dv}{ds}\cos\gamma + \frac{v^2}{r}\cos c \qquad \textbf{(249 c)}$$

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[30] Reech 1852: 265-266.
[31] Reech 1852: 267.

where v is the velocity of a point, r is the radius of curvature of the point's trajectory, s is the direction along the corresponding arc, ,$\alpha\beta$ and γ are the angles between the direction of the point's motion and the x, y and z axes, and a, b and c are the angles between the radius of curvature and these same axes. Reech deduces that the "dynamic forces" vary "partly like the term $\frac{v^2}{r}$ and partly like the term $\frac{v\,dv}{ds}$."[32] It follows that if all homologous velocities have a constant ratio u to the original velocities, the ratio of homologous "forces" varies as the square of the velocities and the inverse of the homologous lengths, . $\frac{u^2}{l}$ [33]

What Reech refers to here as force is in fact acceleration. For homologous "inertial forces" involving mass, he obtains the relation :[34]

$$f = dl^3 \times \frac{u^2}{l} = dl^2u^2. \qquad \textbf{(250)}$$

This law applies to the resistance of an immersed body, provided that it is due solely to inertial forces (as envisaged by Newton). Reech deduced the following law for ships:

> In the absence of the forces of gravity, the resistances of similarly shaped floating bodies with absolutely smooth or polished walls would vary exactly like the densities of liquids, like homologous surfaces and like the squares of velocities."[35]

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[32] Reech 1852: 268-269.
[33] Reech 1852: 270-271.
[34] Reech 1852: 271. Although he is primarily interested in forces, Reech nevertheless indicates a relationship between the durations "of two homologous excursions" as a function of homologous velocity ratios: . $\theta = \frac{l}{u}$
[35] Reech 1852: 272.

Finally, Reech considers the situation where the forces of gravity must be taken into account. In this case, we again have the relationship $f = dl^3$, and since $f = dl^2u^2$, it follows that the condition $u^2 = l$ must be satisfied. [36]

Reech notes:

> The $u^2 = l$ condition will be all the more necessary the more predominant the forces of gravity are in a system, i.e. the slower the motion.[37]

He drew the following conclusion about ship behavior:

> The resistance of a body floating at different speeds and with a suitable pressure at the free surface of the liquid, will have to approach indefinitely the proportionality to the square of the speed, as the movement becomes faster, and will have to move away on the contrary from such a proportionality as the movement becomes slower.[38]

These last two results seem dubious: in fact, gravity forces are responsible for the formation of waves on the water surface in the boat's wake. However, the greater the speed, the greater the resistance due to waves, compared to inertial resistance (due to eddies).

In any case, Reech's dimensional reasoning does imply that a ship whose linear dimensions are l times greater than a model, and which travels at $\sqrt{l}$ times greater speed, will be subjected to l^3 times greater forces (because $f = dl^3$), whose work will be $d\,l^3u = d\,l^2u^3$ greater (because). $u^2 = l$ [39] He points out that this is only true when viscous forces and atmospheric pressure, which do not obey these relationships, can be neglected.[40] Reech has a high opinion of similarity:

> What we have just said sufficiently demonstrates that Newton's theorem on similitude, in mechanics, will always be the best and often the only foundation that

[36] Reech 1852: 272.
[37] Reech 1852: 272.
[38] Reech 1852: 273.
[39] Reech 1852: 273-274.
[40] Reech 1852: 274.

can be given to a host of practical applications.... In particular, this theorem contains almost everything that has so far been established by more or less empirical or faulty means in hydrodynamics.[41]

Newton, Bertrand and Reech's thoughts on similarity were apparently not linked to considerations of homogeneity. Not so in a publication by the great British hydrodynamic master George Gabriel Stokes (1819-1903) from 1850.[42] Stokes considered the damping of the motion of a pendulum immersed in a fluid (air, for example). This led him to study similar pendulums of different sizes. [43]

Stokes starts from the linearized Navier-Stokes equation:

$$\frac{dp}{dx} = \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) - \rho\frac{du}{dt} \tag{251 a}$$

$$\frac{dp}{dy} = \mu\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) - \rho\frac{dv}{dt} \tag{251 b}$$

$$\frac{dp}{dz} = \mu\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) - \rho\frac{dw}{dt} \tag{251 c}$$

and the continuity equation :

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \tag{252 }$$

In these equations u, v and w are the components of the fluid velocity at a point along the x, y and z axes respectively, p is the fluid pressure, ρ its density at the point under consideration and μ is the fluid viscosity.[44]

[41] Reech 1852: 274-275.
[42] Stokes 1850, also in Stokes 1901: 1-141.
[43] Stokes 1901: 16-18.
[44] Stokes 1850, also in Stokes 1901: 13 (definitions of magnitudes 11).

Stokes notes that if a given system satisfies the above equations, a homologous system will also satisfy them provided that

$$u \propto v \propto w, \qquad x \propto y \propto z \qquad \text{and} \qquad p \propto \frac{\mu u}{x} \propto \frac{\rho u x}{t}, \tag{253}$$

which means that the proportions between u, v and w are respected, etc. [45]
It's clear that Stokes obtains the $x \propto y \propto z$ and $p \propto \frac{\mu u}{x} \propto \frac{\rho u x}{t}$ relationships by homogeneity, even if he doesn't use the word and even if there's no question of changing units at all.[46]

The rest of Stokes' discussion concerns pendulums. He uses the symbol a in reference to the linear dimensions of the pendulum globule, and T for a duration concerning it, its period for example. He represents the amplitude of motion of points in the fluid by c, so that $u \propto \frac{c}{T}$. To satisfy geometric similarity, we must have :

$$x \propto a \ , \qquad t \propto T \, . \tag{254}$$

With the condition $\frac{\mu u}{x} \propto \frac{\rho u x}{t}$ or $\frac{x^2}{t} \propto \frac{\mu}{\rho}$, this gives $\frac{a^2}{T} \propto \frac{\mu}{\rho}$, which constitutes a relationship between magnitudes characteristic of pendulum motion and magnitudes specific to the surrounding fluid. Furthermore, the relationship $p \propto \frac{\rho u x}{t}$ implies $p \propto \frac{\rho a c}{T^2}$. Since viscous stresses vary like pressure p (see note on P_1 and T_3), the forces corresponding to them must vary like pa^2, and therefore like . $\frac{\rho a^3 c}{T^2}$

Similar considerations can also be found in the work of English naval engineer William Froude (1810-1879). A former railroad engineer, Froude turned his attention in

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[45] Stokes 1850, also in Stokes 1901: 17.

[46] Indeed, the three terms , $\mu\left(\frac{d^2u}{dx^2}\right) \mu\left(\frac{d^2u}{dy^2}\right)$ and $\mu\left(\frac{d^2u}{dz^2}\right)$ are homogeneous only if , $x \propto y \propto z$ $\frac{dp}{dx}$ is homogeneous with $\mu\left(\frac{d^2u}{dx^2}\right)$ only if $p \propto \frac{\mu u}{x}$, and with $\rho \frac{du}{dt}$ only with $p \propto \frac{\rho u x}{t}$. Stokes 1850: 8, also Stokes 1901: 16.

the 1850s to the study of ship strength. With limited mathematical knowledge, but gifted with a keen physical sense, Froude's main contribution was to make the use of models credible for the study of ship behavior. He achieved this through his intelligent use of the relevant similarity rules.[47]

In this case, his task was to determine the correct relationships between the velocities of a model and a full-size ship, and the corresponding resistances, as a function of the relative dimensions of the two objects. According to Froude, for a quotient λ of the linear dimensions of these, and insofar as only the resistance due to waves (possibly also that due to eddies), the rules are as follows:

$$\frac{V_2}{V_1} = \lambda^{1/2}$$

$$\frac{R_2}{R_1} = \lambda^3$$

where V_1 and V_2 are the speeds of the model and the ship, and R_1 and R_2 the resistances they encounter. [48]

The exact way in which Froude arrived at these results differs from that of Reech. As far as velocities are concerned, he relied on the equation for wave velocity in deep water:

$$V = \sqrt{\frac{gL}{2\pi}}$$

where V is the velocity of a plane wave and L the wavelength. Indeed, if we apply this expression to two systems of different sizes and take the ratio, we obtain :

$$\frac{V_2}{V_1} = \sqrt{\frac{L_2}{L_1}}$$

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[47] Darrigol 2005: 277-283; Wright 1992: 242-254.

[48] Pubflc Record Office (hereafter PRO), ADM 116/137, W. Froude, untitled report to accompany a lost letter dated by other means to 24 April 1868. This is Froude's first report.

However, for the model to represent the behavior of the vessel, the wavelength and velocity of the waves must change scale like the length and velocity of the objects in question; hence the relationship . $\frac{V_2}{V_1} = \lambda^{1/2}$

As far as resistances are concerned, Froude's reasoning can be traced back to another case in which he was more explicit, that of rolling. He states:

> The moment of resistance of the ship, when it performs oscillations of a certain magnitude, exceeds that of the model in a ratio proportional to the fourth power of the scale: since it is formed by the ratio of the square of the velocity of the movement, the areas on which it acts, and the lever with which it acts; the areas are of course proportional to the square of the scale, the lever directly proportional to the scale, and the square of the velocity also directly proportional to the scale, since the period varies as the square root of the scale. [49]

Froude therefore assumes that the moment of resistance obeys the relation :

$$M \,\alpha\, SV^2h$$

where M is the moment of resistance, S the area of the immersed surface, V the linear velocity of the rotational motion, and h the lever. For the velocity, Froude refers to the period, as he assumes that the motion of a pendulum can be used as a model, so :

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Hence : $$\frac{T_2}{T_1} = \lambda^{1/2}$$

For speeds, this means :

$$\frac{V_2}{V_1} = \frac{L_2}{L_1}\frac{T_1}{T_2} = \lambda^{1/2}.$$

[49] "The moment of resistance of the ship, when performing an oscillation of a given range, will, likewise exceed that of the model in the ratio expressed by the fourth power of the scale: since it will be in the compound ratio of the square of the velocity of motion, the areas on which it acts, and the leverage with which it acts; where the areas are, of course, as the square of the scale, the leverage simply as the scale, and the square of the velocity also simply as the scale, since the periodic time is as the square root of the scale." Froude 1863: 271n.

And of course we have : $S = \lambda^2$ and $h = \lambda$.

Therefore : $M = \lambda^4$.

For the case at hand, namely the ratio of resistances due to waves, it's likely that Froude reasoned similarly; the difference being that in this case, since we're dealing with a force and not a moment, the relationship goes as the third power of the linear dimension, not as the fourth power.

We can see that the appeal to dimensional analysis is only implicit in Froude's reasoning. A rigorous derivation of his relations was not what interested Froude: he wanted to show that they were experimentally valid, and his work in this area fulfilled this hope. Despite this, Rayleigh cited Froude as one of the sources of his thoughts on similarity.[50]

8. 4 - Rayleigh's contributions

It wasn't until Rayleigh that the question of similarity was explicitly associated with that of dimensions. It is in the very last pages of his treatise *Theory of sound* that Rayleigh extends dimensional analysis to considerations of similarity, which is indicative of the importance he attaches to these issues. [51]

Although Rayleigh is not very precise in this respect, he seems to have reasoned that in both cases the problem consists in preserving relations between physical quantities while their numerical expression changes - in the first case, this change is due to the change of units, while in the second case it is due to the change of the quantities themselves. Indeed, it will be recalled that in an earlier part of his work, Rayleigh justified the interest of the “method of dimensions” in a manner very similar to Fourier's, by appealing to the invariance of equations under changes of units.[52] However, he makes no mention of such

[50] Rayleigh 1877: vol.2, 288.
[51] Rayleigh 1877: vol.2, 287-289.
[52] See chapter 7, 240-241.

considerations at the end of the treatise, substituting them with considerations of similarity. As a preliminary, he states:

> In the course of this work, we have often had occasion to remark on the importance of the conclusions to which the dimensional method has led us. Now that we are in a position to take examples from a wide variety of acoustic phenomena related to the vibrations of solids and fluids, it is useful to summarize the subject, and to develop in some detail the principles on which it is based.[53]

As an example, he then demonstrates by an argument of dimensional homogeneity (and without reference to the notion of similar systems at this stage) that the period with which an elastic solid vibrates is of the form $\tau \propto q^{-1/2}\rho^{1/2}c$ (where τ is the period, q the Young's modulus, ρ the density and c the linear dimension). Having obtained this result, he explains:

> The argument behind this mathematical shortcut is of the following nature. Imagine two geometrically similar bodies, whose mechanical constitution is the same at their corresponding points, and which execute similar movements in such a way that the corresponding changes take times proportional to their linear dimensions - in the ratio 1 : n, for example. So, if the movement of one of them is possible as a result of elastic forces, so will that of the other.[54]

Rayleigh seems to see similarity as the very reason why dimensional analysis is possible.

Rayleigh shows as follows that similar motions of similar elastic solids satisfy the equations of motion well if times and spaces vary in the same ratio, which he calls "dynamic similarity." Since the linear dimension of the two objects is in the ratio $1:n$, their masses are in the ratio $1:n^3$ (which of course assumes that the two bodies have the same

[53] "In the course of this work we have had frequent occasion to notice the importance of the conclusions that may be arrived at by the method of dimensions. Now that we are in a position to draw illustrations from a greater variety of acoustical phenomena relating to the vibrations of both solids and fluids, it will be convenient to resume the subject, and to develop somewhat in detail the principles upon which the method rests." Rayleigh 1877: vol. 2, 287.

[54] "The argument which underlies this mathematical shorthand is of the following nature. Conceive two geometrically similar bodies, whose mechanical constitution at corresponding points is the same, to execute similar movements in such a manner that the corresponding changes occupy times which are proportional to the linear dimensions - in the ratio, say, of 1 : n. Then, if the one movement be possible as a consequence of the elastic forces, the other will be also." Rayleigh 1877: vol.2, 288.

density). Furthermore, the accelerations of similar points of the two objects are in the ratio $1:n^{-1}$ since the times are in the ratio $1:n$. According to Newton's second law, it follows that the corresponding forces are in the ratio $(1:n^3).(1:n^{-1})=(1:n^2)$, as they should be for elastic forces. In his reasoning, Rayleigh does not spell out the link with dimensions, presumably because he considers it self-evident. For example, in the case of accelerations, we obtain $1:n^{-1}$ by dividing $1:n$ for the linear dimension by $(1:\mathrm{n})^2$ for the square of the time ratio, in the same way as we have . $[\mathrm{a}]=\frac{\mathrm{L}}{\mathrm{T}^2}$

Rayleigh then discusses the case of gravitational oscillations (pendulums and waves on the surface of water), where the forces are proportional to the cube of $1:n$, not its square. Consequently, dynamically similar motions are those for which the ratio of periods is that of the square root of the linear dimensions. Finally, Rayleigh examines the case of sound vibrations, similar to the first case discussed.[55] Since the pressures must be the same, and the forces are pressures multiplied by areas, the forces are in the ratio $1:n^2$, as in the first case. Rayleigh deduces that the systems behave in the same way provided that the ratio of their periods is, like that of their linear dimensions, $1:\mathrm{n}$. Rayleigh deduces that the same conclusions can also be applied to a combination of the first and third cases, for example a tuning fork mounted on a resonant box.[56]

Rayleigh concludes his discussion by extending it to a class of cases in which various spatial dimensions vary in different ratios. He takes as an example an elastic plane

[55] Rayleigh 1877: vol.2, 289.
[56] Rayleigh 1877: vol.2, 289.

whose thickness can vary in different proportions from its width and length. The variables to be taken into account for dimensional analysis are as follows:

- width or length c: L
- thickness b: L
- density ρ : $\mathrm{ML^{-3}}$
- Young's modulus q : $\mathrm{ML^{-1}T^{-2}}$

Since we now have four unknowns, and dimensional considerations offer only three equations, an expression for the vibration period τ cannot be determined from these data alone. Since T only appears in [q], and that to the power -2 , τ must depend on q to the power $-\frac{1}{2}$ for T to be recovered in [τ]. So we have $\tau \propto q^{-1/2}$. Furthermore, $q^{-1/2}$ contributes $\mathrm{M}^{-1/2}$ to the dimensions of τ, which in fact must not contain M. The only quantity that depends on M apart from q is ρ, and we see that it must contribute $\mathrm{M}^{1/2}$to the dimensions of τ. So it must $\tau \propto \rho^{1/2}$. Thus, homogeneity for T and M requires :

$$\tau \propto q^{-1/2}\rho^{1/2} \qquad \textbf{(255)}$$

Since b and c both contain L, we can go no further without additional assumptions. Rayleigh draws on an earlier non-dimensional calculation[57] to admit that the variables q, b and ρ always appeared through the combinations $qb^3$and $b\rho$. He thus obtains:

$$\tau \propto q^{-1/2}b^{(-1/2)3}\rho^{1/2}b^{1/2} = q^{-1/2}\rho^{1/2}b^{-1} \qquad \textbf{(256)}$$

and therefore

$$\tau \propto q^{-1/2}\rho^{1/2}c^2b^{-1}. \qquad \textbf{(257)}$$

The period therefore varies as the square of the length (or width) of the plane if the additional hypothesis is true.

[57] Rayleigh 1877: vol.1, 293-297.

To sum up, the question of similarity in the 19th century has its roots in a theorem stated by Newton, albeit with a very loose interpretation. It wasn't until the 19th century that the question of similarity joined that of homogeneity. This connection is only implicit in some of Stokes' reasoning published in 1850. However, a parallel between changing units and changing scales emerges in the 1840s, with the considerations of Reech and Bertrand: the latter obtain their similarity theorems by requiring the terms of the equation of dynamics to be homogeneous - even though Bertrand sees this as merely an expression of Newton's theorem. Finally, in 1878, Rayleigh explicitly linked similarity to dimensional analysis.

Chapter 9: The Pi theorem

In the early 1890s, a general theorem concerning physical equations appeared: the Pi theorem, also known as the Vaschy-Buckingham theorem. As we shall see, this theorem is not unrelated to the considerations of similarity discussed in the previous chapter. Aimé Vaschy first proposed it in 1892. He drew his inspiration from a work by Emmanuel Carvallo.

9. 1 – Lucas' and Carvallo's curves

French physicist and mathematician Emmanuel Carvallo (1856-1945), whom we already met in Chapter 7 in a dimensional controversy with Clavenad, published the results we're interested in at the end of 1891 and beginning of 1892 in *La lumière électrique*,[1] , citing as his source two papers by Félix Lucas at the Société Mathématique de France, dated December 1891[2] and March 1892.[3] The connection between Lucas' work and dimensional analysis is by no means obvious. Lucas himself claims that it was Carvallo who pointed it out.[4] What Lucas establishes are relationships concerning dynamos, which he expresses in

[1] Carvallo 1891: 505-507; Carvallo 1892c: 554-555; Carvallo 1892b: 21-24.
[2] Carvallo 1891: 506.
[3] Carvallo 1892c: 554.
[4] Lucas 1891: 152.

a form independent of the characteristics of the machine under consideration. He achieves this through a change of variable, the possibility of which Carvallo links to dimensional considerations.

More precisely, Lucas starts from two equations concerning a dynamo placed in a purely resistive circuit.[5] The first expresses the relationship between the average power dissipated in the circuit by the Joule effect, W_m, and the rms intensity of the alternating current I_m:

$$W_m^2 = E_m^2 I_m^2 - \frac{4\pi^2 L^2}{T^2} I_m^4 \tag{258}$$

Where $E_m$is the rms value of the electromotive force, T the rotation period of the alternator, and L its self-induction coefficient.

The second equation expresses the relationship between the average current $I_m$and the inverse of T :

$$I_m^2 = \frac{\dfrac{2\pi^2 Q_0^2}{T^2}}{(\rho + r)^2 + \dfrac{4\pi^2 L^2}{T^2}} \tag{259}$$

where r is the armature resistance, ρ the external circuit resistance, and $Q_0$the maximum value of magnetic flux in the dynamo.

Lucas represents the relationship between the relevant quantities with curves. He first performs a change of variables:

$$x = \frac{2\pi L}{E_m T} I_m , \qquad y = \frac{2\pi L}{E_m^2 T} W_m . \tag{260}$$

[5] Lucas 1892: 404-406.

Equation (1) becomes : $$y^2 = x^2 - x^4. \qquad (261)$$

For the second equation, he defines the variables :

$$x = \frac{2\pi L}{(\rho + r)} \frac{1}{T},\, ,\, y = \frac{L\sqrt{2}}{Q_o} I_m \qquad (262)$$

which leads to equation : $$x^2 y^2 - x^2 + y^2 = 0. \qquad (263)$$

Lucas' only comment is that "the fourth-degree curve that this equation represents is, to within a few scales, the intensity curve under consideration,"[6] as well as a similar comment for the first equation. There is nothing to suggest that he had conceived his approach as anything other than a simple change of variables prior to Carvallo's intervention.[7] Carvallo took Lucas's reference to scales very seriously,[8] , and set about generalizing his colleague's results through a study based on dimensional analysis. He imagines two machines of the same type, both characterized by the same kinds of constants (resistance, inductance, etc.), and involving the same physical variables. He assumes that, as in the cases treated by Lucas, these physical variables are two in number, and designates them X and Y, since they constitute the coordinates of the curves to be represented. He designates the characteristic constants of the first machine as A_i , and the corresponding constants of the second machine as B_i . An equation such as those discussed by Lucas can therefore be represented for the first machine by :

$$f(X, Y, A_1, A_2, \ldots) = 0, \qquad (264)$$

and for the second by :

$$f(X, Y, B_1, B_2, \ldots) = 0. \qquad (265)$$

[6] Lucas 1892: 406.
[7] Lucas 1891: 152-154.
[8] See the quote given a few lines above.

Carvallo then imagines a change of units for the latter, and represents the numerical values of X, Y, B_1, B_2etc. in this new system by the corresponding lower-case letters, so that the equation becomes :

$$f(x, y, b_1, b_2, \ldots) = 0 \,. \qquad \textbf{(266)}$$

If there are no more than three characteristic constants, Carvallo deems it possible to choose the three fundamental units of the L, M, T system in such a way that , $b_1 = A_1$ $b_2 = A_2$, and so on. Equations (7) and (9) are then "identical,"[9] except that the two variables are given in different unit systems. As Carvallo says,

> This change in units is reflected graphically by a change in the abscissa and ordinate scales of the characteristic curve.[10]

Carvallo concludes the following general theorem:

> If the characteristic equation of a machine type contains no more than three characteristic constants, the characteristic curves of the various machines of this type can be deduced from each other by a simple change in the abscissa and ordinate scales.[11]

The situation can be represented by the following diagram, where each of the two curves corresponds to a machine with different characteristics:

[9] Carvallo 1891: 507.
[10] Carvallo 1891: 507.
[11] Carvallo 1891: 507.

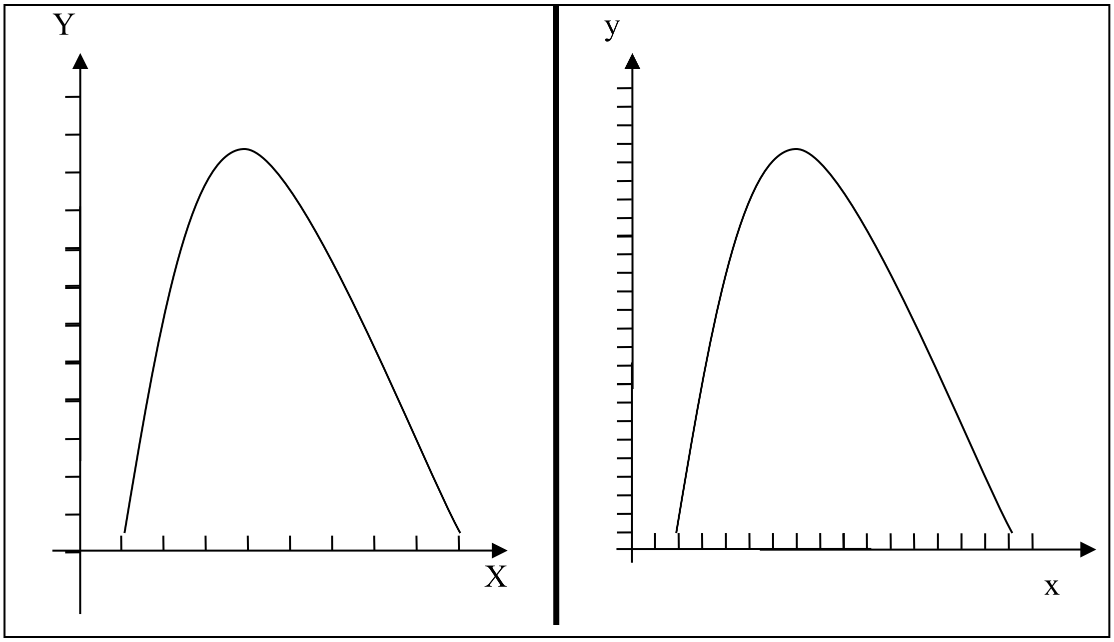

Figure 9.1

That the number of characteristic constants should not exceed three is obviously a necessary condition if their values are to be reduced to any values given by a change in the three units M, L, T. Shortly afterwards, Carvallo realized that this condition was not sufficient, and he added:

> The measurements of the three characteristic constants must be *independent* functions of the fundamental units of length, time and mass.[12]

He offers a counter-example to illustrate this point.[13] The following differential equation relates to alternators:

$$I + \frac{L}{R}\frac{dI}{dt} - \frac{E_0}{R}\sin\frac{2\pi t}{T} = 0, \qquad \textbf{(267)}$$

[12] My italics. Carvallo 1892b: 21-24; Carvallo 1892c: 554. Carvallo gives this restriction, accompanied by an example, as early as his publication in *Les Annales Télégraphiques*. But he didn't demonstrate it in detail until his article in *La lumière électrique*.
[13] Carvallo 1892b: 23-24.

where I is the current intensity at time t, $E_0$the maximum induction electromotive force, L the self-induction coefficient, R the armature resistance and T the period. The machine's three characteristic constants are $,\frac{L}{R}\frac{E_0}{R}$ and $\frac{2\pi}{T}$. Of course, the latter has the dimension T^{-1}; but $\frac{L}{R}$ has the dimension T, since RI must be homogeneous with $L\frac{dI}{dt}$. So we can't obtain an abstract equation.

In a later article,[14] Carvallo takes account of the independence condition and modifies his initial reasoning: rather than talking about a change in the units M, L T, he reasons on the basis of a change in fundamental quantities. He now considers a single equation, valid for a machine with characteristic constants $,\,,A_1\ A_2\ A_3$:

$$f(X, Y, A_1, A_2, A_3, \ldots) = 0. \tag{268}$$

These three constants have dimensional formulas in the L, M, T system:

$$[A_1] = \mathrm{L}^{\lambda_1}\mathrm{M}^{\mu_1}\mathrm{T}^{\tau_1}\ ,\,,. \quad [A_2] = \mathrm{L}^{\lambda_2}\mathrm{M}^{\mu_2}\mathrm{T}^{\tau_2} \quad [A_3] = \mathrm{L}^{\lambda_3}\mathrm{M}^{\mu_3}\mathrm{T}^{\tau_3} \tag{269}$$

Due to the assumed independence of the variations of the three constants when changing units L, M, T, it is possible to define three new independent units $,\,,\mathrm{A}_1\ \mathrm{A}_2\ \mathrm{A}_3$ by , , $\mathrm{A}_1 = \mathrm{L}^{\lambda_1}\mathrm{M}^{\mu_1}\mathrm{T}^{\tau_1}\ \mathrm{A}_2 = \mathrm{L}^{\lambda_2}\mathrm{M}^{\mu_2}\mathrm{T}^{\tau_2}\ \mathrm{A}_3 = \mathrm{L}^{\lambda_3}\mathrm{M}^{\mu_3}\mathrm{T}^{\tau_3}$ and invert these relations to obtain :

$$\mathrm{L} = \mathrm{A}_1^{\lambda'}\mathrm{A}_2^{\lambda''}\mathrm{A}_3^{\lambda'''}\ \mathrm{M} = \mathrm{A}_1^{\mu'}\mathrm{A}_2^{\mu''}\mathrm{A}_3^{\mu'''}\ \text{ and } \mathrm{T} = \mathrm{A}_1^{\tau'}\mathrm{A}_2^{\tau''}\mathrm{A}_3^{\tau'''}$$

[14] It would seem that Carvallo's presentation was inspired by Lucas' questions. Indeed, he introduces his discussion by saying "Here is how we can establish the general proposition in the form desired by M. Lucas." Carvallo 1892c: 554.

It is therefore possible to express the variables X and Y in the units $A_{(1)\,A1}$, $A_{(2)\,A2}$, $A_{(3)\,A3}$, if we know the formulas for the dimensions of X and Y in the L, M, T system and therefore those in the A_1, A_2, A_3system. The latter formulas are given as

$$[X] = \mathrm{A}_1^{\alpha_1}\,\mathrm{A}_2^{\alpha_2}\,\mathrm{A}_3^{\alpha_3} \quad \text{and} \quad [Y] = \mathrm{A}_1^{\beta_1}\,\mathrm{A}_2^{\beta_2}\,\mathrm{A}_3^{\beta_3}, \tag{270}$$

we have :

$$X = xA_1^{\alpha_1}\,A_2^{\alpha_2}\,A_3^{\alpha_3} \quad \text{and} \quad Y = yA_1^{\beta_1}\,A_2^{\beta_2}\,A_3^{\beta_3} \tag{271}$$

If x and y are the new variables, the new characteristic equation of the machine is equation :

$$f(x, y, 1, 1, 1) = 0 \tag{272}$$

because the homogeneity of an equation implies that a relationship between measurements of various

quantities retain the same form in any system of units. [15]

We can therefore see that the equation is the same for all possible choices of characteristic constants, with the exception of a homothety of the machine variables. Equation (16) embraces Lucas' equations (4) or (6) obtained by changing variables. Lucas calls this the "abstract form" of the machine characteristic equation. Similarly, Carvallo calls equation (16) "the abstract equation." From the latter, we can recover the "concrete equation" of a given machine by reverting to the units M, L, T.[16]

Carvallo is aware that he is dealing with a question of similarity, as he uses the term in the title of his publications. Admittedly, we're not talking here about the effect of a change in the size of a physical system, but of a change in other characteristic quantities. However, when these are multiplied by suitably chosen constants, the fundamental

[15] Carvallo 1892c: 555.
[16] Carvallo 1892b: 24.

equations remain the same. This is a generalization of similitude. Moreover, we can see that Carvallo takes advantage of the link between similarity in this generalized sense (change of scale in the broad sense), and change of units, although he does not comment explicitly on this point.

9. 2 - Vaschy's generalization: The Pi theorem

In January 1892, in the same issue of *Annales Télégraphiques* as Carvallo's work discussed above,[17] Vaschy proposed his famous theorem.[18] It's clear that he draws on Carvallo's discussion and theorem, which he mentions as a special case of the one he offers. Vaschy regards his own theorem as very general, and even says that "the most general law of similitude in mechanics and physics" results from it.[19] He formulates it as follows:

> Let $a_1, a_2, a_3, \ldots a_{\mathrm{n}}$ be physical quantities, the first p of which are related to distinct fundamental units and the last $(n-p)$ to units derived from the p fundamental units (e.g. a_1 can be a length, a_2 a mass, a_3 a time, and the $(n-3)$ other quantities $a_4, a_5, \ldots a_{\mathrm{n}}$ would be forces, velocities, etc.; then p = 3). If between these n quantities there is a relationship :
> $$\mathrm{F}(a_1, a_2, a_3, \ldots a_{\mathrm{n}}) = 0,$$
> which remains whatever the arbitrary quantities of the fundamental units, this relationship can be reduced to another between $(n-p)$ parameters at most, i.e. :
> $$f(x_1, x_2, \ldots x_{n-p}) = 0,$$
> the parameters $x_1, x_2, \ldots x_{n-p}$ being monomial functions of $a_1, a_2, \ldots a_{\mathrm{n}}$ (for example $x_1 = \mathrm{A}\, a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_n^{\alpha_n}$).[20]

Monomial functions are dimensionless here.

[17] This article follows Carvallo's.
[18] Vaschy 1892a: 25-28.
[19] Vaschy 1892a: 25.
[20] Vaschy 1892a: 25.

This theorem generalizes Carvallo's in two ways. Firstly, Vaschy is interested not just in machines, but in any type of physical system. He's not talking about characteristic constants (of a machine), but more generally about physical quantities. The distinction he makes is not between variables and characteristic constants, but between physical quantities corresponding to fundamental quantities on the one hand, and derived quantities on the other. Moreover, as is clear from the examples he discusses below, Vaschy considers the choice of fundamental quantities to be conventional in the application of his theorem. However, he considers their number to be well-defined in a given situation.

Secondly, from a mathematical point of view, Vaschy generalizes Carvallo's theorem by assuming an arbitrary number (*n*) of physical quantities and fundamental units (*p*). Vaschy himself says: "In the particular case where $n = 5$ and $p = 3$, we find the theorem given by Mr. Carvallo."[21] The stylistic difference between the two theorems lies in the fact that this particular case allows representation in the form of a curve. Let's recall how Carvallo formulates his theorem:

> If the characteristic equation of a machine type contains no more than three characteristic constants, the characteristic curves of the various machines of this type can be deduced from each other by a simple change in the abscissa and ordinate scales.[22]

The essence of Carvallo's theorem is that a change in the scale of variables can absorb the dependence of the characteristic equation on the characteristic quantities, provided the latter are no more numerous than the fundamental quantities (necessarily three in Carvallo's case). The essence of Vaschy's theorem is that an equation can be reduced to a number

[21] Vaschy 1892a: 25.
[22] Carvallo 1891: 507.

$n-p$ of variables (or "parameters"), without explicit dependence on the fundamental quantities. In both cases, this is only possible by redefining the units, fundamental quantities or variables (Lucas), whereby the additional information is implicitly included in the variables or parameters remaining in the equation.

The form of Vaschy's theorem suggests that he was familiar with Carvallo's demonstration of March 1892, either in its published form or as a possible earlier version.[23] Indeed, Vaschy's definition $x_1 = A\,a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_n^{\alpha_n}$ of "parameters" is probably inspired by the way Carvallo redefines his variables ($X = A_1^{\alpha_1}\, A_2^{\alpha_2}\, A_3^{\alpha_3} x\ Y = A_1^{\beta_1}\, A_2^{\beta_2}\, A_3^{\beta_3} y$). We can see that even some of the symbols used are similar, despite differences in interpretation. In his demonstration, Vaschy points out that, since the physical quantities $a_{p+1}, \ldots a_n$ are derived quantities, they can be expressed in terms of the fundamental quantities at various powers, multiplied by a numerical factor. This is equivalent to saying that these quantities satisfy relations of the type :

$$\frac{a_{p+1}}{a_1^{\alpha}\, a_2^{\beta} \ldots a_p^{\lambda}} = x_1\ ,, \quad \frac{a_{p+2}}{a_1^{\alpha'}\, a_2^{\beta'} \ldots a_p^{\lambda'}} = x_2 \qquad \textbf{(273)}$$

etc. up to a_n, where $x_i$are numerical values *independent of the choice of fundamental units*. Vaschy gives as an example a situation where the derived quantity $a_4$is a force (and the fundamental quantities length (a_1) , mass (a_2) and time (a_3) are always present); the ratio $\frac{a_4}{a_1\, a_2\, a_3^{-2}}$ is then independent of units.

[23] Carvallo 1891: 505-507.

The fact that we can rewrite $a_{(p+1)} \ldots a_n$ as a function of $a_1, a_2, \ldots a_p$ implies that the initial equation

$$F\left(a_1, a_2, \ldots a_p, a_{p+1}, a_{p+2}, \ldots\right) = 0 \qquad \textbf{(274)}$$

can be written as :

$$F\left(a_1, a_2, \ldots a_p, x_1\, a_1^{\alpha} \ldots a_p^{\lambda}, x_2\, a_1^{\alpha'} \ldots a_p^{\lambda'}, \ldots\right) = 0\,, \qquad \textbf{(275)}$$

or by homogeneity,

$$F\left(1, 1, \ldots 1, x_1, x_2, \ldots x_{\mathrm{n-p}}\right) = 0. \qquad \textbf{(276)}$$

This means that there is a function f of $n - p$ variables such that the initial equation reduces to :

$$f\left(x_1, x_2, \ldots x_{n\text{-}p}\right) = 0. \qquad \textbf{(277)}$$

Vaschy's reasoning, like Carvallo's, involves taking units equal to the quantities to be eliminated.

Vaschy's work is also reminiscent of Bertrand's in 1878, and to a lesser extent of Rayleigh's in the 1877 *Theory of sound* (from which Bertrand was probably inspired.)[24] John Roche sees Vaschy's work as the most significant consequence of Rayleigh's, though he is careful not to speculate on how this inspiration came about. In all these cases, the starting point is an equation containing the expression of a physical quantity as a function of the other quantities on which it may depend. Bertrand and Vaschy then form 0-dimensional expressions of the latter. For Vaschy, the function in question is abstract, representing the relations between the quantities on which it depends, and the right-hand side of the equation is 0. With Rayleigh and Bertrand, we're dealing with a physical

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[24] See chapter 7, p.251.

quantity (a duration in his example), which is a function of other quantities. Vaschy's work can therefore be seen as a rigorous development of Bertrand's, with his theorem explaining and justifying the latter's method. It is conceivable that Vaschy was inspired by Bertrand's reasoning, but he mentions neither Bertrand nor Rayleigh.[25] He only quotes Carvallo. As we have seen, Carvallo was probably inspired by Rayleigh.[26] There is therefore a more or less direct link between Rayleigh and Vaschy.[27]

Vaschy concludes his discussion with two applications. The first concerns the oscillations of a pendulum.[28] The relevant physical quantities are period *T*, length *l*, mass *m*, gravitational acceleration *g*, and angular amplitude α. The starting equation is :

$$F\left(T, l,\ m,\ g, \alpha\right) = 0. \tag{278}$$

The fundamental quantities are the three usual mechanical quantities, to which *T*, *l* and *m* correspond here. In the previous notation :

$$a_1 = T;\ \ a_2 = l;\ \ a_3 = m;\ \ a_4 = g;\ \ a_5 = \alpha \qquad \text{with } n = 3$$

The next step is to express a_4 and a_5 in terms of the other quantities. This requires knowledge of their dimensional formulas. As $a_5 = \alpha$ is dimensionless, Vaschy keeps it as it is. Since the quantity $a_4 = g$ is an acceleration of dimensions LT^{-2}, Vaschy posits $\frac{gT^2}{l} = x_1$.

The theorem then yields

$$f\left(x_1, \alpha\right) = 0, \tag{279}$$

what Vaschy writes:

[25] See chapter 7.
[26] See chapter 7, 251 & 258.
[27] Roche 1998: 209.
[28] Vaschy 1892a: 26-27.

$$T = \sqrt{\frac{l}{g}}\varphi(\alpha), \quad \textbf{(280)}$$

where $\varphi(\alpha)$ is dimensionless.

The second application given by Vaschy concerns the electric current intensity at time *t* at the end of a telegraph cable, which has been connected to the battery at time . $t_0 = 0$ [29] The physical quantities to be considered are time *t*, the length *l* of the cable, its capacitance *C*, its resistance *R*, and the electromotive force *E* of the battery. Vaschy therefore starts from equation :

$$F\left(t, l,\ C,\ E, R, i\right) = 0, \quad \textbf{(281)}$$

or even

$$\Phi\left(t, l,\ C, CE^2,\ R,\ Ri\right) = 0. \quad \textbf{(282)}$$

His fundamental quantities are time *t*, length *l*, capacity *C* and energy $C\,E^2$. Having $n = 6$ physical quantities in total and $p = 4$ fundamental quantities, Vaschy must have $n - p = 2$ derived quantities, which must give rise to numerical expressions independent of the choice of units. As elementary electrical circuit theory suggests, these expressions must be $R\frac{C}{t}$ and $\frac{Ri}{E} = Ri\sqrt{\frac{C}{CE^2}}$. The reduced equation is

$$i = \frac{E}{R}\varphi\left(\frac{t}{C\,R}\right). \quad \textbf{(283)}$$

As Vaschy points out, this equation conforms to the law discovered by William Thomson. He stresses that his reasoning is not based on the choice of an electrostatic or

[29] Vaschy 1892a: 27-28.

electromagnetic system: the expressions he chooses are dimensionless, whatever the system considered.

A few months later, Vaschy devoted a long article to various other applications of his theorem.[30] The vast majority of these applications concern well-known physical phenomena, but the last two have a more ambitious theoretical scope. Vaschy begins by justifying his use of four fundamental quantities in electricity, due to the uncertainty concerning the dimensions in terms of M, L and T of the constants in the electrostatic and magnetic Coulomb formulas: he therefore suggests the use of an additional fundamental quantity corresponding to one or other of these constants. [31]

In his first example, Vaschy considers the discharge of a capacitor through a resistor, and is interested in the potential difference V at time t after the capacitor has begun to discharge.[32] This value depends on the resistance R of the circuit, the capacitance C of the capacitor and its potential difference V_0 at time : $t_0 = 0$

$$V = \varphi\left(V_0, t, C, R\right), \qquad \textbf{(284)}$$

he rewrites:

$$V = V_0\,\psi\left(t, C, C V_0^2, \frac{t}{CR}\right), \qquad \textbf{(285)}$$

where ψ must be dimensionless. Vaschy takes the fundamental quantities t, C and $C V_0^2$.

Since $\frac{t}{CR}$ is dimensionless, he obtains :

$$V = V_0\,\psi\left(\frac{t}{CR}\right).^{33} \qquad \textbf{(286)}$$

[30] Vaschy 1892b: 189-211.
[31] Vaschy 1892b: 190-194.
[32] Vaschy 1892b: 199-200.
[33] This ψ function is of course different from the previous one. Vaschy 1892b: 200.

Note that this reasoning is a shortcut to the reasoning used in the illustrations in the previous article, as it immediately extracts the variable of interest.

Vaschy then discusses current quenching in an inductive and resistive circuit that is disconnected from a battery.[34] The relevant quantities are the circuit's resistance *R*, its self-induction *L*, the initial current $i_0$and the current *i* at time *t*. We have :

$$i = \varphi\left(i_0, R, L, t\right), \tag{287 }$$

Vaschy rewrites:

$$i = i_0\,\psi\left(t, R, R\,i_0^2 t, \frac{R\,t}{L}\right). \tag{288 }$$

Whereψ is dimensionless. In addition to *t*, Vaschy takes the energy $R\,i_0^2 t$ and the resistance *R* as fundamental quantities. Since $\frac{R\,t}{L}$ is dimensionless, he obtains :

$$i = i_0\,\psi\left(\frac{R\,t}{L}\right) \ . \tag{289 }$$

Vaschy also considers the case of oscillations in an *LC* circuit. The main variable is the current *i* at time *t*.[35] The other relevant quantities are the self-induction of the conductor *L*, the capacitance of the capacitor *C*, and its potential difference V_0. We have

$$i = \varphi\left(V_0, C, L, t\right), \tag{ 290}$$

Or :

$$i = V_0\sqrt{\frac{C}{L}}\,\psi\left(C, \sqrt{CL}, C\,V_0^2, \frac{t}{\sqrt{CL}}\right). \tag{291 }$$

[34] Vaschy 1892b: 202-203.
[35] Vaschy 1892b: 203-204.

Vaschy takes as fundamental quantities $C, \sqrt{CL}$ which has the dimensions of time, and $C V_0^2$ which has those of energy. As $\frac{t}{\sqrt{CL}}$ is dimensionless,

$$i = V_0 \sqrt{\frac{C}{L}} \psi \left(\frac{t}{\sqrt{CL}} \right). \tag{292}$$

In other cases, the function ψ depends only on quantities that can be considered fundamental, and is therefore reduced to a numerical constant.

For example, Vaschy considers the speed υ of propagation of the first traces of current in a telegraph cable.[36] This depends on the line's specific resistance ρ, its specific capacitance γ, and its self-induction λ :

$$\upsilon = \varphi\left(\rho, \gamma, \lambda \right), \tag{293}$$

that Vaschy rewrites:

$$\upsilon = \frac{1}{\sqrt{\gamma\lambda}} \psi \left(\rho, \rho\,\gamma, \frac{1}{\sqrt{\gamma\lambda}} \right) = 0\,, \tag{294}$$

where $\frac{1}{\sqrt{\gamma\lambda}}$ has dimensions , LT^{-1} $\gamma\rho$ has dimensions $\mathrm{L}^{-2}\mathrm{T}$. This allows Vaschy to treat them as independent fundamental quantities. As for ρ, according to Vaschy its dimensions are not reducible to those of mechanics, and it can constitute a separate fundamental quantity. Vaschy's theorem therefore implies that ψ reduces to a numerical constant (A):

$$\upsilon = \frac{A}{\sqrt{\gamma\lambda}}. \tag{295}$$

[36] Vaschy 1892b: 196-198.

A similar situation arises when Vaschy deals with the period T of oscillations in an LC circuit.[37] As before, the relevant quantities are the self-induction of the conductor L, the capacitance of the capacitor C and its potential difference : V_0

$$T = \varphi\left(V_0, C, L\right), \tag{296}$$

which becomes :

$$T = \sqrt{CL}\,\psi\left(C, \sqrt{CL}, C V_0^2\right), \tag{297}$$

where ψ now depends only on quantities that can be considered fundamental. Vaschy concludes:

$$T = A\sqrt{CL}\,, \tag{298}$$

where A is a numerical constant.

The above examples concern well-known physical phenomena. But Vaschy also draws more theoretical consequences from his theorem: he demonstrates that certain electromagnetic interactions cannot exist, or that there are restrictions on their form. First, he envisages a force between a quantity of electricity q and a quantity of magnetism μ.[38] This hypothetical force f could depend only on q, μ , the distance r between the two, and the coefficients k and k' of Coulomb's laws for electrostatics and magnetism :

$$f = \varphi\left(q, \mu, r, k, k'\right). \tag{299}$$

Since $\sqrt{kk'}\,\dfrac{q\,\mu}{r^2}$ has the dimensions of a force, Vaschy obtains :

$$f = \sqrt{kk'}\,\frac{q\,\mu}{r^2}\,\theta\left(r, k, k'\right). \tag{300}$$

[37] Vaschy 1892b: 203.
[38] Vaschy 1892b: 206-210.

As $\sqrt{kk'}$ has the dimensions of a velocity, *r*, *k* and *k*' can be treated as fundamental quantities. The function θ can therefore be reduced to a numerical constant and :

$$f = A\sqrt{kk'}\,\frac{q\,\mu}{r^2}. \tag{301}$$

Using non-dimensional reasoning that we won't reproduce here, Vaschy shows that *A* is equal to one. However, a force of an intensity corresponding to (45) in this case should have been observed if it existed.[39]

The second case considered by Vaschy is that of a force *f* between a quantity of electricity *q* and a circuit element of length *ds* through which a current *i* flows.[40] The other relevant variables are the distance *r* between *q* and this circuit element; the angle ε that *ds* makes with the straight line that joins its center to *q*; and *k* and k' :

$$f = \varphi\left(q,i,ds,r,\varepsilon,k,k'\right). \tag{302}$$

An expression with dimensions equal to those of a force is $\sqrt{\frac{k}{k'}}\frac{q\,i\,ds}{r^2}$, hence

$$f = \sqrt{\frac{k}{k'}}\frac{q\,i\,ds}{r^2}\theta\left(r,\varepsilon,k,k'\right) \tag{303}$$

The variables *r*, *k* and k' can again be treated as fundamental quantities, but ε, being an angle, is of zero dimension. Therefore :

$$f = \sqrt{\frac{k}{k'}}\frac{q\,i\,ds}{r^2}\chi\left(\varepsilon\right). \tag{304}$$

This time, Vaschy goes no further: "We can't see how, without arbitrary assumptions, we could determine either the function $\chi\left(\varepsilon\right)$, or the very direction of the force *f* ".[41]

[39] Vaschy 1892b: 209.
[40] Vaschy 1892b: 211.
[41] Vaschy 1892b: 210-211.

A theorem very similar to Vaschy's was proposed some 15 years after his death, in 1914, by the American physicist Edgar Buckingham.[42] The statement is as follows:

> Let's describe by means of an equation a relationship existing between several physical quantities of *n* kinds. If several quantities of the same kind appear in this relationship, let's specify them by the value of one of them and the quotients of the others to that one. The equation then contains *n* symbols $\mathrm{Q}_1,...\mathrm{Q}_\mathrm{n}$, one for each kind of quantity, and also, in general, several quotients *r′*, *r,″* etc., and can be written as $f(Q_1, Q_2,...Q_n, r', r'',...) = 0$. Let's assume for the moment that the quotients *r* remain constant during the course of the phenomenon described by the equation: for example, if the equation describes a property of a material system that includes lengths, the system will remain geometrically similar to itself during any change in sizes. Under this condition, [the above equation] reduces to $F(Q_1, Q_2,...Q_n) = 0$ .... Due to the homogeneity principle, [any equation of this form] is reducible to the form [$\psi(\Pi_1, \Pi_2,...\Pi_\mathrm{i}) = 0$] where . $[\Pi_1] = [\Pi_2] = ... = [\Pi_\mathrm{i}] = [1]$[43]

Buckingham then shows that $i = n - k$, where *k* is the number of fundamental units.

Buckingham's theorem is therefore virtually identical to Vaschy's. Where Vaschy gave himself the relation $\mathrm{F}(a_1, a_2, a_3,...a_n) = 0$, Buckingham uses the formula $f(Q_1, Q_2,...Q_n, r', r'',...) = 0$; and where Vaschy arrived at $f(x_1, x_2,...x_{n-p}) = 0$, Buckingham arrives at $\psi(\Pi_1, \Pi_2,...\Pi_i) = 0$, where *i* is equal, as with Vaschy, to the number of arguments in the starting equation minus the number of fundamental units. In both cases, the

[42] Buckingham 1914: 345-376.

[43] "Let it be required to describe by an equation, a relation which subsists among a number of physical quantities of *n* different kinds. If several quantities of any one kind are involved in the relation, let them be specified by the value of any one and the ratios of the others to this one. The equation will then contain *n* symbols $Q_1,...Q_n$, one for each kind of quantity, and also, in general, a number of ratios *r′*, *r,″* etc., so that it may be written: $f(Q_1, Q_2,...Q_n, r', r'',...) = 0$. Let us suppose, for the present only, that the ratios *r* do not vary during the phenomenon described by the equation: for example, if the equation describes a property of a material system and involves lengths, the system shall remain geometrically similar to itself during any changes of size that may occur. Under this condition [the above equation] reduces to $F(Q_1, Q_2,...Q_n) = 0$ .... By reason of the principle of dimensional homogeneity, every complete physical equation of [this form] is reducible to the form [$\psi(\Pi_1, \Pi_2,...\Pi_\mathrm{i}) = 0$] in which $[\Pi_1] = [\Pi_2] = ... = [\Pi_\mathrm{i}] = [1]$. Buckingham 1914: 345-348.

arguments that appear in the resulting equations, x and Π, are dimensionless. The only notable differences are that Buckingham mentions deals with the case where several of the Q quantities happen to be of the same type, and the fact that Buckingham's article details how to obtain the dimensionless expressions, which Vaschy left implicit.

The similarities between Buckingham's work and that of Vaschy suggest that they are not independent, although Buckingham makes no reference to Vaschy in his publication. This theorem is now known as the Pi theorem (due to the fact that the parameters are products, as expressed by the letter designating them in Buckingham's article). But as Enzo Macagno notes:

> If the Pi theorem is to be attributed to someone, it should certainly be called the Carvallo-Vaschy theorem or simply the Vaschy theorem, and Buckingham's name should simply be associated with the presentation of a new proof.[44]

As for Roche, he writes: "In the form later developed by the American physicist Edgar Buckingham, this theorem [by Vaschy] became known as the Pi theorem." [45]

The theorem proposed by Vaschy is an important discovery: today, the Pi theorem is a basic theorem of dimensional analysis. Its origins lie in a series of generalizations. In their seminal work, Lucas and Carvallo both confined themselves to machines. Lucas performs changes of variables to show that, in specific cases, the equations governing the two machine variables are the same to within one change of scale, whatever the machine's characteristic constants. Carvallo generalized the discussion to any pairs of variables and triplets of characteristic constants, and explained Lucas' scaling relations first by changing

[44] "If the π-theorem has to be associated with a personal name it should certainly be called the theorem of Carvallo-Vaschy or simply the theorem of Vaschy, and the name of Buckingham should be associated only with the presentation of a new proof," Macagno 1971: 400.
[45] "In its further development by the American physicist Edgar Buckingham, this theorem became known as the Pi theorem," Roche 1998: 210.

units and then, in a later work, by changing fundamental quantities. In a more formal approach, Vaschy generalizes Carvallo's results to arbitrary systems and an arbitrary number of quantities.

As Macagno notes, Vaschy's theorem had no immediate impact.[46] It was only in the twentieth century, after being rediscovered in 1914 by Buckingham, that it aroused the interest of physicists and engineers, particularly in fluid dynamics.

[46] Macagno 1971: 398.

Chapter 10: The physical meaning of dimensions - the nature of quantities and physical models

10. 1 - Introduction

We have seen that in the last decades of the 19th century, dimensional analysis was used in an innovative way, leading to the discovery of new relationships between physical quantities through various lines of reasoning: Rayleigh's “method of dimensions”, similarity considerations and Vaschy's theorem. Apart from this fertility, what role could dimensional analysis have played in understanding phenomena? To answer this question, two aspects need to be considered: firstly, the conception of the physical quantities themselves, and secondly, the models designed to explain the phenomena.

As far as the first point is concerned, the question that arises is: what link do we see between the dimensions of a quantity and its physical nature? Let's remember that before Fourier, with a few exceptions, the concept of dimension was, so to speak, synonymous with that of the nature of a quantity. The very term "homogeneity" bears witness to this meaning: etymologically, "homogeneous" means "of the same kind."

As we saw in Chapter 6, Mercadier and Vaschy share this view. In their 1883 memoir, they asserted:

> The dimensions of a quantity result... from the nature of this quantity and are in a way its analytical representation. [1]
>
> If, moreover, as a result of progress in science, the quantity in question were recognized to be of a different nature from that which had been attributed to it at the outset, we would, it is true, be obliged to change its dimensions; but it would then be a different quantity from the previous one, and we would have to give it a different name. The other would remain what it was, with its original definition and dimensions.[2]

In 1890, Vaschy wrote:

> Let's start by noting that the addition or comparison of two quantities only makes sense if these quantities are of the same nature and, consequently, have the same dimensions. You can't add a length to a time, or a mass to an acceleration. On the other hand, the square of a speed is comparable or homogeneous to the product of an acceleration and a length.[3]
>
> Considered as units, the quantities L, M, T, A, F,... refer to standards of length, mass... Considered as dimensions, they refer to the mechanical or physical nature of the quantities in question· [4]

Vaschy even thought that a dimension formula uniquely characterized a physical quantity. Let's recall what he said in this regard:

> Generally speaking, specific dimensions can only relate to a specific physical quantity, and are sufficient to determine its nature, if not to explain its inner workings.[5]

The position defended by Mercadier and Vaschy was original at the time: due to the development of dimensional analysis in the context of changes of units in the period preceding their work, their predecessors found nothing shocking in the fact that the same quantity had a certain dimensional formula in one "system of units" and a different formula

[1] Mercadier & Vaschy 1883b: 6.
[2] Mercadier & Vaschy 1883b: 6. See also Mercadier 1883a: 71.
[3] Vaschy 1890: 3.
[4] Vaschy 1890: 4.
[5] Vaschy 1883a: 47.

in another. To what extent has Vaschy's and Mercadier's work changed perceptions of the link between the dimensions of a quantity and its nature? Do we now consider, as they did, that the nature of a physical quantity determines its dimensions? Conversely, will dimension formulas be perceived as translating the physical nature of quantities?

10. 2 - Does the nature of a quantity determine its dimensions?

The notion that a physical quantity has only one dimensional formula seems to have been widely adopted. As we mentioned in Chapter 6, the work of Mercadier and Vaschy is uncritical in this respect; and it is subsequently found in several authors. For example, Lucas uses it as an argument in favor of using magnetic mass as the fourth fundamental unit:

> A magnetic potential, whose symbolic unit is $L^2MT^{-2}\mu^{-1}$, and a magnetic power, whose symbolic unit is $L^{-1}\mu$, are two quantities of different natures; but when we replace μ by $L^{3/2}M^{1/2}T^{-1}$, the symbolic units of magnetic potential and magnetic power become both $L^{1/2}M^{1/2}T^{-1}$ so that they become identified, as if they referred to two quantities of the same nature.[6]

Augusto Rovida declared the following year, 1894:

> Isn't this *infinity of absolute* systems of dimensions absurd?
> It seems to me that the most elementary logic should lead us to the unification of systems. And what does "absolute system of dimensions" really mean? We have to admit that natural quantities have ties and relationships with each other and with the fundamental quantities - length, mass and time - that are completely outside the scope of the contributions that the human mind can make to them; they are therefore absolute links that must be manifested by a unique dimension for each physical quantity given and formed with the help of the fundamental quantities.[7]

[6] Lucas 1893: 392.
[7] Rovida 1894: 606.

The question of the uniqueness of the dimensions of physical quantities is often linked to the Gaussian mechanistic position that any physical quantity can be reduced to the units of the fundamental quantities of mechanics.[8] Vaschy and Mercadier themselves refer to dimensions in the system M, L, T. But this mechanistic conception is not obvious to everyone. For example, in 1896, in his attempt to determine the dimensions of the coefficients k and k'[9] as a function of the fundamental quantities L, M and T,[10] Joubin began by admitting that he wasn't sure it was possible:

> There is no *a priori* indication that these quantities can be expressed exclusively in mechanical units, i.e. that electrical or magnetic phenomena are manifestations of purely mechanical properties of a medium.[11]

Taking the mechanistic position as his working hypothesis, Joubin expresses it as follows:

> Electrical and magnetic quantities are of the same nature as those of rational mechanics.[12]

In his mind, the mechanistic postulate is therefore inseparable from the problem of the nature of magnitudes.

However, the idea that the nature of a quantity defines its dimensions is not unanimously accepted. A particularly instructive case is that of J.J. Thomson. In 1897, he declared:

> It's probably necessary to make it clear from the outset that the "dimensions" of electrical quantities are a matter of definition, and depend entirely on the system of units we adopt.... In fact, we could choose a system of units in such a way as to give any dimensions to any electrical quantity. [13]

[8] See chapter 2, 64.
[9] The coefficients of the electrostatic and magnetic Coulomb formulas.
[10] See chapter 6, 229.
[11] Joubin 1896a 188-191; also published under the same title in *Joubin 1896b:* 398.
[12] Joubin 1896a 188-191; also published under the same title in *Joubin 1896b:* 399.
[13] "It may be well to state at the outset that the "dimensions "of electrical quantities are a matter of definition and depend entirely upon the system of units we adopt. Thus we shall find that on the electromagnetic system of units a resistance has the same dimensions as a velocity, while on the electrostatic system of units it has

Of course, one might assume that he is merely describing an unfortunate state of affairs, but this is not the case. Thomson distinguishes between concrete quantities and their "properties":

> We need to fix electrical quantities by one or other of their properties. So, to take an example, we can fix an electric charge by the repulsion it exerts on an equal charge, as is done in the electrostatic system of units, or by the force on a magnetic pole when the charge is transferred from one place to another by a current, as is the case in the electromagnetic system; these two measurements are of different dimensions. To take a simpler example, we could "fix" a quantity of water by the number of hydrogen atoms it contains, by its mass, or by its volume at a given temperature; all these measures would be of different dimensions.[14]

In the contexts Thomson deals with, dimensions are associated with the characteristic *properties* of a quantity. But the quantity of electricity is not a property. It therefore makes no sense to attribute definite dimensions to it. Although we often refer to the concrete quantity and one of its characteristic properties by the same expression as for a volume, for example, i.e. "physical quantity" or "physical magnitude", we are dealing with two different things. In Thomson's view, the nature of the electric charge does not determine its dimensions. It's hard to imagine a more profound disagreement with the views expressed some fifteen years earlier by Mercadier and Vaschy.

Thomson's views are linked to a conception of dimensions as being associated with the act of measurement. They assume that the formulas for the dimensions of quantities are

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the same dimensions as the reciprocal of a velocity. In fact we might choose a system of units so as to make any one electrical quantity of any assigned dimensions; when the dimensions of this are fixed that of the others becomes quite determinate." Thomson J.J. 1897a: 446.

[14] "We have to fix the electrical quantities by one or other of their properties. Thus, to take an example, we may fix a charge of electricity by the repulsion it exerts on an equal charge, as is done in the electrostatic system of units, or by the force experienced by a magnetic pole when the charge is being transferred from one place to another by a current, as is done in the electromagnetic system; these two measures are of different dimensions. To take a simpler case we might fix a quantity of water by the number of hydrogen atoms it contains, by its mass, or by its volume at a definite temperature; all these measures would be of different dimensions." Thomson J.J. 1897b: 447.

the expression of the absolute measurements defined by Gauss and Weber: these formulas express how the measurement of the quantity "reduces" to measurements of length, time and mass. In Thomson's mind, dimensions cannot express nature: they are quantitative, not qualitative.

In France, a similar concept can be found in Gabriel Lippmann's work. In 1887 - four years after the series of publications by Mercadier and Vaschy - Lippmann set out to define a new time standard, based on electrical phenomena.[15] Lippmann's article on the subject illustrates the need for caution when interpreting certain statements. In the context of this study, he first defines the time standard by the "specific resistance... of mercury in the electrostatic system."[16] Indeed, in this system, resistance has dimensions $\frac{\mathrm{T}}{\mathrm{L}}$ (see chapter 3), and by "specific resistance" Lippmann means resistance per unit length (of mercury, for example); it therefore has dimensions T.[17] Lippmann thus equates resistivity with duration, for dimensional reasons.[18] He even goes so far as to state:

> [Specific resistance] ρ is not simply a quantity whose measurement is related to the measurement of time: it is a concrete time interval, disregarding any measurement convention or choice of unit.[19]

[15] Lippmann 1887a: 1070-1074; also published under the same title in Joubin 1896b*:* 2ème série, vol.6, 1887, 261-265.
Lippmann notes that the second, and any other unit that would be a fraction of a natural cycle such as the year, is not in fact stable, but varies when sufficiently long durations are considered. He therefore suggests defining a new unit of time independent of such cycles, and, in this case, using electrical phenomena. This new unit would be the time required for a unit quantity of electricity to pass through a circuit subjected to a unit electromotive force, when the resistance of this circuit is equal to that of a mercury cube with sides equal to the unit length. (Lippmann 1887a: 1070-1071).
[16] Lippmann 1887a: 1071.
[17] More precisely: "If we consider a circuit having a resistance equal to that of a cube of mercury whose side would have the unit of length, a circuit subjected to an electromotive force equal to the unit, this circuit will take to let itself be crossed by the unit of quantity of electricity a determined time, which is precisely ρ." Lippmann 1887a: 1071.
[18] In fact, he says: "It should be noted that the choice of the unit of length, like that of the unit of mass, is indifferent; for the various units brought into play here depend on them in such a way that ρ *does* not depend on them." and he specifies in a note "In other words, ρ is of the first degree with respect to time, of the zero degree with respect to the other units." Lippmann 1887a: 1071.
[19] Lippmann 1887a: 1071.

Despite this remark, we must beware of assuming that Lippmann equates the dimensions of a quantity with its nature in a general way. He explains further: [20]

> We know that some of the particularly simple dimensional formulas used in electricity give the idea of a corresponding physical interpretation. Thus, capacitance, expressed in absolute electrostatic units, has the dimensions of length; electrical resistance, in absolute electromagnetic units, has the dimensions of velocity. It's often said that electrical capacitance is a length, resistance a speed. This is obviously inaccurate. The nature of an electrical quantity cannot change with the conventions used to express it numerically, and which determine the dimensional formula.[21]

It should be noted that language shortcuts will survive, but they will generally be perceived as such and will hardly cause confusion.[22]

Lippmann's criticism of a naïve physical interpretation of the dimensions attributed to electrical quantities seems to echo that of Mercadier.[23] In fact, Lippmann's conceptions are far removed from those of Mercadier and Vaschy: for him, dimension formulas are due

[20] Lippmann 1887b: 638-640.
[21] Lippmann 1887b: 638.
[22] Hospitalier (probably Edouard Hospitalier (1853-1907), Professor at the École de Physique et de Chimie industrielles de la Ville de Paris) said: "In all the works of Lord Kelvin - an undisputed authority in England and throughout the world - aren't resistances expressed in centimeters per second? Doesn't Dr. Hopkinson express torque in joules in his studies of tramways? Lord Kelvin knows that resistance is not speed, and Dr Hopkinson knows that torque is not work. They use these units, however, and rightly so, because they are the result of assumptions made in order to establish a coherent system that is as simple as possible, and which is naturally affected by these assumptions. The day when a more advanced state of science will enable us to do away with these hypotheses and establish the absolute dimensions of physical quantities, but only that day, will we have to start all over again and create new names in greater numbers, since it will no longer be legitimate to measure with the same unit two quantities that do not have the same dimensions," Hospitalier 1896: 340.
[23] Mercadier writes: "In the electrostatic system the capacitance has the dimensions of a length; in the electromagnetic system it has the dimensions of the inverse of an acceleration." We're saying [here] something that isn't precisely inaccurate, in the sense that there are *implied conventions* that may allow us to express ourselves in this way, but it certainly is if we want to consider the substance of the things we're talking about. It would be much more correct, much more exact, to modify the proposition as follows: 'In the electrostatic system, the capacitance has the dimensions of a length multiplied by the inverse of the coefficient of Coulomb's law; in the electro-magnetic system, it has the dimensions of the inverse of an acceleration multiplied by the inverse of the coefficient of Ampère's law'. With this last statement, we understand that we're dealing with the *same* quantity expressed in two ways that may be *different* only *in appearance* because of the existence of coefficients." Mercadier 1883a: 71.

to conventions, and their only interest is that they facilitate conversion calculations.[24] Like Thomson, he sees dimensions as intimately linked to a measurement protocol:

> Capacitance in absolute electrostatic units has the dimensions of a length: this means that we can experimentally find the numerical expression of an electrical capacitance without measuring any quantity other than a length. In other words, using operations that don't involve any kind of measurement, we can construct a length such that all we need to do is measure it to obtain the desired number.[25]

It deals with electrical resistance in the same way.[26]

10. 3 - Do the dimensions of a quantity reflect its nature?

The perception that the dimensions of a quantity shed light on its nature seems to have been less widespread. Many scholars express the opposite view. Their position is that the dimensions of a quantity offer only partial information about it. Lippmann, for example, is convinced of the futility of attempts to better understand electrical quantities on the basis of their dimensions:

> In general, no electrical quantity, and no expression formed with these quantities, seems susceptible to physical interpretation, whatever its dimensional formula.[27]

More subtly, he writes:

> The dimension formula provides...the necessary, but not sufficient, condition for the possible physical interpretation.[28]

A similar opinion can be found among other scholars. Lodge wrote in 1888:

> The mere dimensions of a quantity do not always determine its nature. For example, the moment of a force has the dimensions of work, yet it is not work, and cannot

[24] "This one [the dimension formula] has only one meaning and one object: it serves to calculate the function by which the numerical expression of a quantity must be multiplied when the fundamental units are changed." Lippmann 1887b: 638.
[25] Lippmann 1887b: 638.
[26] Lippmann 1887b: 639.
[27] Lippmann 1887b: 640.
[28] Lippmann 1887b: 638.

> enter into an equation that includes terms representing work. Furthermore, the circular measure of an angle is not a pure number, although it has no dimensions like a pure number; and that it is not a pure number is physically obvious, since a moment of force × an angle = work.[29]

Lodge's arguments have nothing to do with the possibility or otherwise of reducing electromagnetic quantities to mechanical ones, since the quantities he is dealing with here are all mechanical. What they do illustrate is that the process by which the dimensions of a physical quantity are determined involves a loss of information about that quantity in relation to the equation that defines it.

This loss of information is not always the reason why some scientists are pessimistic about the ability of dimensions to convey the nature of physical quantities. In 1897, J.J. Thomson declared, with reference to electrical quantities (his statement could be applied to derived quantities in general):

> A symbol representing an electrical quantity only tells us how much of the quantity there is, and tells us nothing about the nature of the quantity; that would require a dynamic theory of electricity. A theory of magnitudes cannot tell us what electricity is; its purpose is simply to enable us to find the change in the numerical measurement of a certain electric charge or other electric quantity when the units of length, mass and time are altered in a specified way.[30]

Thomson's comparison with the symbol representing the quantity of electricity is revealing: he believes that dimensions in particular, like symbols representing quantities in general, are concerned with *quantities*, not *qualities*. This dichotomy reflects the dilemma that has

[29] "Merely the dimensions of a quantity do not always fix the kind of quantity. For example, the moment of a force is of the dimensions of work, and yet it is not work, and cannot exist as a term in an equation involving *work* terms. Again, the circular measure of an angle is not a pure number, though it is of zero dimensions as a pure number is; and that it is not a pure number is evident physically, for a moment of a force × an angle = work." Lodge 1888: 282.

[30] "A symbol representing an electrical quantity merely tells us how much of the quantity there is, and does not tell us anything about the nature of the quantity; this would require a dynamical theory of electricity. A theory of dimensions cannot tell us what electricity is; its object is merely to enable us to find the change in the numerical measure of a given charge of electricity or any other electrical quantity when the units of length, mass and time are changed in any determinate way." Thomson J.J. 1897a: 447.

arisen in the interpretation of dimensions since Fourier: as mentioned above, on the one hand, the very notion of dimension was born out of the desire to respect the principle of homogeneity, and thus to reflect the nature of quantities**.** On the other hand, since Fourier, this notion has been defined via the effect of a change of units on the numerical value of quantities, i.e. in quantitative terms.

Nevertheless, some scientists believe that the dimensions of a quantity shed light on its nature. Would they go so far as to think that they totally determine it? For some, this seems to be the case. Of course, the idea that dimensions inform us about the nature of quantities is often only implicit. However, three explicit statements can be cited. In 1896, Joubin asserted that determining the dimensions of k and k' "would be of such great interest, and of such a nature as to throw such a bright light on a whole class of phenomena as yet so unknown."[31] At the same time, Rovida asserted:

A dimensional quantity is essentially physical: it must exist in nature.[32]

Vaschy, in his treatise, notes that the fact that the product of the coefficients of the electrostatic and magnetic Coulomb formulae, kk', has a velocity as its dimension inspired Maxwell to develop his electromagnetic theory of light; in this case, Vaschy seems to believe that the dimensions of kk' define its nature: [33]

> In electricity, if we take the product kk' of the coefficients of the fundamental formulas, we see that it has the dimensions of the square of a velocity; we have numerically found for the value of this product the square of the speed of propagation of light in a vacuum. Maxwell deduced that electrical or electromagnetic vibrations must propagate in the ether like luminous vibrations and with the same speed, thus establishing a new link between optical and electrical phenomena, which would have been difficult to suspect without the help of absolute measurements.[34]

[31] Joubin 1896a 188-191; also published in Joubin 1896b: 398.
[32] Rovida 1894: 602.
[33] Vaschy 1883b: 16.
[34] Vaschy 1890: 16.

The idea that the dimensions of a physical quantity reflect its nature underpins certain attempts at interpretation, and even certain models. Nevertheless, such attempts are very rare, probably because for the phenomena that raised the most questions in the 19th century, namely electromagnetic phenomena, the very determination of the dimensions of quantities is problematic. I present below a sample of these few attempts.

In an article published in 1883 - at the same time as his seminal paper with Mercadier - Vaschy explores the problem of the nature of electricity using dimensional considerations: [35]

> Without resorting to hypotheses, we can, by simply relying on the consideration of dimensions, seek to conceive the physical nature of the quantities we encounter in the theory of electricity.[36]

His starting point is Coulomb's formula, which he gives in the following form:

$$f = V^2 \frac{qq'}{r^2}, \tag{305}$$

where V^2 is the square of a velocity, as he opts for the electromagnetic system. The "surface electrical density" δ (charge per unit area), Vaschy continues, has dimensions $\mathrm{L}^{-3/2}\mathrm{M}^{1/2}$, which are also those of the square root of a "*material mass* density" (his italics).[37] For this reason, Vaschy identifies the quantity δ^2 with a density of matter, which he designates ρ. He then reminds us that the electrostatic pressure at the surface of an electrified conductor is given (in the electromagnetic system) by :

$$p = 2\pi\delta^2 V^2, \tag{306}$$

[35] Vaschy 1883b: 182-185.
[36] Vaschy 1883b: 182.
[37] Vaschy 1883b: 182.

or :

$$p = 2\pi \rho V^2 . \tag{307}$$

Vaschy was struck by the similarity of this formula to the

$$p = \frac{1}{3} \rho V^2 \tag{308}$$

giving the pressure of a gas in molecular kinetic theory as a function of its density ρ and the root-mean-square velocity V^2 of its molecules. He speculates that these two formulas represent phenomena similar in their mechanisms:

> In both cases, V represents the translational velocity of molecules or atoms in the surrounding medium, this medium being the ether in the case of an electrified body immersed in a vacuum. In both cases, ρ represents a material mass density. This density, it's true, is well-defined and, as it were, visible in the second case, whereas in the electrostatic phenomenon we don't see the matter whose density is represented by ρ or by ρ^2. But the very comparison I have just made between these two orders of phenomena could perhaps serve to clarify the meaning of density ρ and help to conceive the mechanism of electrical actions.[38]

Vaschy's suggestion is bold, but remains rather indefinite as to the details of the mechanism involved. He even hesitates between ρ and ρ^2 for the density of ethereal matter.[39] As we know, electrostatic pressure is always directed outwards from an electrified conductor. This difference from the pressure exerted on a body immersed in a gas is not lost on Vaschy. He offers the following explanation: in the case of a gas, we're dealing with molecules that are too large to penetrate the body's interior, whereas in the case of electrical phenomena, we're dealing with ether atoms that are small enough to penetrate the body's interior:

> This is not a fundamental difference[40] and does not prove that the explanation of electrostatic facts should be sought elsewhere than in a comparison with the

[38] Vaschy 1883b: 183-184.

[39] Perhaps he has in mind Laplace's theories where pressure is proportional to the square of density. Chang 2004: 72.

[40] Vaschy is referring here to the fact that a gas can only compress, whereas electrified bodies can expand.

> pressure of a fluid medium on the bodies it bathes. It simply shows that the electrostatic problem is more complex than the other, and requires a less summary solution.[41]

Vaschy is well aware that dimensional considerations play a crucial role in the reasoning that led him to propose his kinetic model:

> The search for *an* absolute system of electrical units is of even greater interest than fixing the dimensions of electrical quantities. It would constitute a very important step towards the discovery of the intimate mechanism of electricity. It would highlight the role of the ethereal agent in electrical phenomena, and provide a natural link between Coulomb's law and Ampère's law. Perhaps it would even lead to a true physical explanation of the phenomena and an a *priori* demonstration of Coulomb's law. [42]

However, dimensional considerations are not enough. They are complemented by an analogy with a mechanical model. And Vaschy is undoubtedly aware of the heuristic character of his approach, since he does not hesitate to mention the possibility of another mechanical model of electrical actions in which electrified bodies vibrate within an incompressible continuous fluid, following an idea by Carl Anton Bjerknes. [43]

A few years later, on the other side of the Channel, FitzGerald also linked dimensional analysis to a mechanical theory of the ether. In 1885, he conceived of the ether as a perfect liquid traversed in all directions by filamentary vortices ("vortex sponge").[44] In 1889, he presented dimensional considerations in line with such a model: he proposed a hybrid system of units in which electric and magnetic capacitance both have the dimensions of the inverse of a velocity (see Chapter 4). He deduces that these quantities are "of the nature of a slowness." And he adds:

[41] Vaschy 1883b: 184.
[42] Vaschy 1883a: 50.
[43] Vaschy 1883b: 185. Bjerknes had presented demonstrations to this effect two years earlier, at the 1881 Electricity Exhibition in Paris. Forbes 1881.
[44] Darrigol 2000: 189 and Darrigol 2005: 242-243.

> It seems possible [that the capacities] are related to the inverse of the square root of the average energy of turbulence in the aether.[45]

Finally, an 1894 article by Augusto Rovida even attempts to clarify the nature of mechanical quantities on the basis of their dimensions. Although Rovida's reasoning is dubious and confused, it does present some interesting aspects from our point of view. In accordance with Newton's second law, Rovida attributes the dimensions $L^3M^{-1}T^{-2}$ to the gravitational constant. He interprets L^3M^{-1} as "specific volume" (i.e. volume per unit mass), and associates T^{-2} with "acceleration" (in fact, with a second derivative with respect to time). He sees this as an argument in favor of the notion that the sun's thermal energy comes from a change in gravitational potential energy due to the star's contraction. Although unfortunate, this attempt is interesting in that Rovida attempts to use dimensional analysis to determine the physical significance of quantities and to guide the search for a model of a still unexplained phenomenon.[46]

To sum up, after Vaschy and Mercadier's 1883 publications, it is commonplace to see the assertion that the nature of quantities determines their dimensions. To a lesser extent, some physicists use dimension formulas to clarify or even define the physical nature of the corresponding quantities. This second register includes wild speculations such as those of Rovida, but also more coherent attempts such as those of Vaschy himself and FitzGerald.

[45] "It seems possible that they [the capacities] are related to the reciprocal of the square root of the mean energy of turbulence of the aether." Fitzgerald 1889: 323.
[46] Rovida 1894: 603.

Conclusion

The 19th century saw the development of dimensional analysis in the modern sense of the term. Its antecedents can be found in the requirements of the homogeneity principle, which states that certain operations (notably addition and subtraction) can only be performed on homogeneous quantities. Such considerations date back to antiquity. At the time, they concerned only geometric quantities, and the term dimension was understood in this sense. In the 17th century, Descartes and Wallis used the term dimension to designate quantities other than geometric ones, and even negative dimensions. Newton, for his part, developed the concept of derived quantities. Earlier, at the end of the 16th century, Viète had proposed some algebraic rules to ensure the homogeneity of equations. The 18th century saw few innovations concerning homogeneity. The development of a numerical conception of equations seemed to make it possible to sidestep the question. The issue reappeared at the beginning of the 19th century. Fourier took an interest in homogeneity as early as 1811, and Poisson even used it for demonstration purposes in 1816, to obtain the relationship between the speed of a wave and other physical quantities.

It was in 1822 that the concept of dimension really took on its modern meaning, when Fourier linked it to the problem of conversions between systems of units in his treatise *Théorie de la chaleur*. Fourier equated the requirement for homogeneity of equations with that of invariance by change of unit. He defined the concept of dimension exponent as the power to which the conversion factor of a derived quantity must be set when a fundamental unit is changed. The need to extend the concept of dimensions to quantities other than length thus naturally arises: a dimension exponent must be taken into account for each quantity considered as fundamental (in Fourier's case: length, time and temperature). While remaining within the context of the validity of equations, Fourier redefined the concept of "dimension. The latter is decoupled from the notion of the nature of a physical quantity; it is now defined through the intermediary of measurement (via the problem of conversion between units of measurement). From an epistemological point of view, Fourier's gesture can be interpreted as a positivist anticipation.

It's by no means clear that Fourier's new definition is compatible with the earlier one, according to which the dimension of a quantity characterized its nature. Throughout the 19th century, the tension between these two conceptions of homogeneity can be found, sometimes explicitly, more often unconsciously.

Fourier's work had no immediate impact.

From the 1830s to the 1850s, there were a number of works which, while not explicitly involving the concept of dimension, were close to Fourier's in their interest in units. This is particularly true of the work of Gauss and Weber in Germany. These two scientists were convinced of the need to define new, stable units for electrical and magnetic quantities. In describing methods for converting from one system to the other, they arrived at considerations close to those of Fourier. Since they assumed that electromagnetic quantities could all be expressed in terms of mechanical quantities, they defined an "absolute unit" for each quantity, derived from the so-called fundamental mechanical units of length, time and mass. In the 1840s and 1850s, Weber defined several distinct systems of absolute units, including the so-called "electrostatic" and "electromagnetic" systems, which came into widespread use in the decades that followed.

From the 1860s onwards, certain aspects of Fourier's work became increasingly popular. Fourier himself attached great theoretical importance to the subject of dimensions: for him, they were what enabled the then numerical equations to make sense. His readers were more interested in the practical aspect, i.e. the conversion calculations facilitated by Fourier's definition of dimensions. The need for such calculations increased with the development of electric telegraphy. Against this backdrop, the electromagnetic system was adopted in 1863 by a committee of the British Association for the Advancement of Science. James Clerk Maxwell and telegraph engineer Fleeming Jenkin dealt explicitly with dimensional considerations in the committee's report. Their work represents a kind of synthesis between the conceptions of Gauss and Weber on the one hand, and those of Fourier on the other. Like their British colleagues on the committee, they share with the former the mechanistic view that measurements of derived quantities can be reduced to length, time and mass. From Fourier, they took over the concept of dimension defined in the context of changes of units, and in particular the concern to determine to what power the units of fundamental quantities intervene in those of derived quantities.

Ten years later, in 1873, Maxwell's treatise *Treatise on Electricity and Magnetism* was published, and was to have a considerable influence not only in Great Britain, but also in France. Maxwell included a systematic exposition of Fourier's theory of dimensions, and gave the dimensions of electrical and magnetic quantities in various systems. Taking as his starting point fifteen dimensional equations considered fundamental, he deduced dimensional formulas independent of the unit system considered for each of the quantities he dealt with. Only later did he derive the electrostatic and electrodynamic systems, assigning different dimensions to charge and magnetism in each.

Presumably as a result of the popularity of this treatise, dimensional analysis spread quite rapidly, and a great deal of thought was devoted to the subject. Some of these reflections concern the foundations of dimensional analysis, and look at possible alternatives to the use of length, time and mass as fundamental quantities: to reduce one of these to the other two (Everett, Géraldy), to reduce them to one (Weber, Szavardy), to add others (Rücker, Lucas, Vaschy, Bertrand, Mercadier), or to change their nature (Everett proposes length, time and density; Clavenad proposes length, time, and electric charge; Malagoli proposes electric charge, quantity of magnetism and work). The motivations of these authors are diverse. They may be linked to energetic conceptions, or to aesthetic criteria, in particular the desire to obtain a system in which the exponents of dimensions are simpler, notably integers (Everett, Malagoli). These various suggestions were not widely adopted.

In the second half of the 1870s in Great Britain and in the 1880s in France, dimensional analysis made its appearance in physics textbooks. This type of dissemination is also linked to the question of electrical and magnetic units and the multiplicity of systems in use. Despite the *British Association* Committee's choice of the electromagnetic system, the electrostatic system survived for practical reasons linked to telegraph line capacitance measurements. As the concept of dimension is thus linked to those of unit system and conversion, it seems natural that the same physical quantity should have different dimensional formulas in the two systems.

But this diversity can be a source of embarrassment, as illustrated by a controversy in Great Britain in 1882 over the dimensions of the magnetic pole. The trigger was a paper in which Clausius proposed different dimensions for the magnetic pole in the electrostatic system than those given by Maxwell. What was at issue was the formula for the dimensions of a quantity in the *same* system of units. Two strategies were developed to resolve the issue: on the one hand, an attempt was made to establish a hierarchy between the equations from which the dimensions were obtained; on the other - and this was the most successful approach - the formula used by Clausius was considered incorrect and corrected by introducing a coefficient to reflect the influence of the environment. The controversy came to an end with a contribution by Lodge, in which the divergence in dimension formulas was attributed to a difference between Continental and British scientific cultures: the former tended to ignore the environment, whereas in the latter it appeared to be of primary importance.

The following year, in 1883, the notion that a physical quantity could legitimately possess different dimensional formulas in different systems of units was vigorously challenged. Going back to the idea that dimensions reflect the nature of physical quantities, two telegraph engineers, Ernest Mercadier and Aimé Vaschy, found the existence of several dimensional systems theoretically unacceptable. They attributed this ambiguity to ignorance of the true dimensions of the coefficients used in the electromagnetic equations: the electrostatic system treated the coefficient of Coulomb's electrostatic formula as dimensionless, and the electromagnetic system treated the coefficient of the equivalent equation for magnetic poles and that of Ampère's formula in the same way; but these were arbitrary choices. Mercadier and Vaschy set out to demonstrate that, in reality, only the assumptions of the electromagnetic system are likely to be correct, and they relied on experiment to confirm this. The principle of their experiments is to prove that the coefficients of the magnetic and electromagnetic force laws do not depend on the medium (assumed to be non-magnetic). They believe they can conclude that these coefficients are dimensionless, because they admit that if a quantity is a constant independent of any controllable physical circumstance, it is necessarily dimensionless.

This last point earns them immediate criticism. The long-term impact of their intervention is more difficult to assess. The two engineers did not put an end to the debate on the plurality of unit systems in electromagnetism. However, many French people seem to share their judgment: this multiplicity is unsatisfactory, there is in fact only one legitimate dimension formula for each physical quantity, and the plurality that prevails stems from ignorance of the dimension formulas for the coefficients involved. Some assume this ignorance by adding new fundamental quantities to length, time and mass. Others try to make up for it with theoretical arguments - rather than experimental ones, as suggested by Mercadier and Vaschy. Still others, including Mercadier himself in the early 1890s, attempted to establish dimensional relationships that were valid whatever the system under consideration, as Maxwell had already done in his Traité. Mercadier went so far as to redefine the concept of system, basing it on the notion of dimensions rather than units.

The question of units and systems in electromagnetism was not the only one to be agitated in the last third of the 19th century. Other works involved original uses of dimensional analysis. As early as 1871, Rayleigh used dimensional analysis to obtain new relationships between certain physical quantities. In his 1877 treatise *Theory of sound*, he turned it into a veritable "dimensional method": he showed how, knowing the quantities to be taken into account and their dimensional formulas, it was possible in certain cases to establish the equation linking these physical quantities. Shortly afterwards, examples of similar reasoning appeared in France. In 1892, a controversy arose over the speed of signal propagation in telegraph cables. In addition to the weakness of the reasoning of one of the protagonists, this debate illustrates the power and limits of the dimensional method. As Rayleigh pointed out as long ago as 1877, one of these limits is the need to know in advance all the variables on which the quantity of interest is likely to depend, as well as their dimensions.

Theory of sound tackles another problem linked to dimensional analysis: similarity. Rayleigh sees this as the reason for the power of his "dimensional method": from the formal point of view of equations, the changes of units of dimensional analysis are equivalent to the scale changes of similitude. Indeed, in both cases, the aim is to preserve the validity of relationships between physical quantities as their numerical expression changes - due to a change of units in the case of dimensional analysis, due to a change in the quantities themselves in the case of similitude.

Some fifteen years later, a succession of dimensional considerations metamorphosed a work on dynamos by engineer Félix Lucas into an important theorem of dimensional analysis, "Vaschy's theorem." At the end of 1891, all Lucas had in mind was a change of variables to absorb the characteristic constants of dynamos. Emmanuel Carvallo soon generalized Lucas' work with a theorem that could be applied to any machine. Vaschy's theorem appears as a further generalization of Carvallo's work: it concerns any physical system, not just machines, and involves any number of quantities (whereas Carvallo considered three characteristic constants and two physical variables). This theorem makes it possible to reduce the number of quantities involved in an equation by grouping them together in the form of dimensionless monomials. It is based on similarity, in the sense that it allows systems of different scales (in the broadest sense) to be represented by the same equation, but also on dimensional analysis, since its demonstration and application essentially involve the dimensions of the different quantities and the homogeneity of the equation linking them.

In addition to these more formal developments in dimensional analysis, there are a number of attempts to use it to determine the nature of physical quantities and the relevant physical models. The most successful example is Rayleigh's 1871 explanation of the blue of the sky by diffusion due to atmospheric molecules. The other attempts are linked to Vaschy's idea that dimensions reflect the nature of a quantity. Vaschy was one of those who sketched a model of the electromagnetic ether based on dimensional analogies with other branches of physics.

However, the perception that the dimensions of a quantity determine its nature is not unanimous. As Lodge noted in 1888, even among mechanical quantities, quantities of different natures can have the same dimensional formula. The process by which the latter is determined implies a loss of information about the physical quantity in question, in relation to the equation that defines it. Vaschy's strong realist conception didn't last long. On the other hand, his idea, already present in Fourier's work, that a given physical quantity can only have one formula of dimensions, is found in later works.

The question of whether dimensions are an intrinsic characteristic of quantities or simply of their measurement cuts across the problem of the relationship between quantities and measurement, and, beyond the purely historical approach we have chosen, raises philosophical issues. As Gauss and Weber clearly saw, the dimensions of a quantity are associated with the measurement process that links them to the fundamental quantities (time, space and mass, for example). The question then arises as to whether knowledge of this process alone is sufficient to determine the quantity and its dimensions.

If this is the case, two measuring devices associated with formulas of different dimensions must necessarily correspond to two different quantities. We would have to say, for example, that electrostatic charge and electrodynamic charge are two distinct quantities, differing from each other by a factor c that has the dimension of a velocity. Such a situation would resemble an instrumentalist approach in which physical quantities are entirely defined by the concrete operations that link them to fundamental quantities.

On the other hand, if it is possible to define an ontology of quantities that transcends their knowledge through measurement, the measurement process is not sufficient to determine the latter. In this case, even when different experimental processes lead to formulas of different dimensions, the latter may correspond to the same quantity. This is because dimensioned coefficients can be included in the laws linking the quantity under consideration to the fundamental quantities. For example, the coefficients of Coulomb's and Ampère's laws can be given dimensions such that the charge quantity has the same dimension in the expression of the electrostatic and electrodynamic forces that enable measurement. This conception has often gone hand in hand with a reductionist approach. Vaschy and Mercadier, for example, believed that a mechanistic reduction of electromagnetic phenomena would remove the ambiguity of dimensions. In any case, the internal coherence and univocity of the theory's empirical predictions are preserved.

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